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A SIMPLE PERMUTOASSOCIAHEDRON

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Abstract. In the early 1990s, a family of combinatorial CW-complexes named permutoassociahedra was introduced by Kapranov, and it was realised by Reiner and Ziegler as a family of convex polytopes. The polytopes in this family are "hybrids" of permutohedra and associahedra. Since permutohedra and associahedra are simple, it is natural to search for a family of simple permutoassociahedra, which is still adequate for a topological proof of Mac Lane's coherence. This paper presents such a family.

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The paper is dedicated to Jim Stasheff whose work is the cornerstone of this beautiful mathematics.

1. Introduction

A geometry of commutativity and associativity interaction is demonstrated by polytopes called permutoassociahedra. This family of polytopes arises as a "hybrid" of two more familiar families—permutohedra and associahedra. Kapranov, [13], defined a combinatorial CW-complex KP_n (denoted by $\mathbf{KPA}_{n-1}$ in [22]), whose vertices correspond to all possible complete bracketings of permuted products of n letters. He showed that for every n, this CW-complex is an $(n-1)$-ball, and found a three dimensional polytope that realises KP_4. Reiner and Ziegler, [22], realised all these CW-complexes as convex polytopes.

However, for every $n \geq 4$, the polytope KP_n is not simple. Our goal is to define a new family of polytopes $\mathbf{PA}_{n-1}$, having the same vertices as KP_n, which are all simple, and which may still serve as a topological proof of Mac Lane's coherence.

The combinatorics of the family $\mathbf{PA}_n$ is introduced in [19]. This family is called *permutohedron-based associahedra* in that paper, and for the sake of simplicity, we call it here *simple permutoassociahedra*. From [19, Proposition 10.1], it follows that there exist simple polytopes that realise these combinatorially defined objects.

Our polytopes $\mathbf{PA}_n$ belong to a family that generalises polytopes called *nestohedra* or *hypergraph polytopes*. The work of Fulton and MacPherson, [8], De Concini and Procesi, [3], Stasheff and Shnider, [24], Gaiffi, [9], [10], Feichtner and Kozlov, [6], Feichtner and Sturmfels, [7], Carr and Devadoss, [2], Postnikov, Reiner and Williams, [21], Postnikov, [20], Došen and the third author, [5], and of many others established the family of nestohedra as an important tool joining algebra, combinatorics, geometry, logic, topology and some other fields of mathematics.

We pay attention in particular to 0, 1 and 2-faces of polytopes $\mathbf{PA}_n$. These faces suggest a choice of generators and corresponding equations relevant for building a symmetric monoidal category freely generated by a set of objects. All this is important for Mac Lane's coherence. However, the main result of the paper is an explicit realisation of the family $\mathbf{PA}_n$ given by systems of inequalities representing halfspaces in $\mathbf{R}^{n+1}$. Our realisation has its roots in the realisation of associahedra and cyclohedra given in [24].

Throughout the text, the subset relation is denoted by $\subseteq$, while the *proper* subset relation is denoted by $\subset$.

2. Varying the combinatorics by generators

In this section we try to formalise the main differences between the approach of [13], leading to the family of permutoassociahedra KP_n, and our approach, leading to a family of simple permutoassociahedra. Roughly speaking, the main difference is the choice of arrows that generate symmetry in a symmetric monoidal category. The reader interested mainly in combinatorics of the new family of polytopes, may skip this section and just take a look at the commutative diagrams given below.

The notion of symmetric monoidal categories (not under this name) is given, and a coherence result is proven by Mac Lane in [15]. (For the definition of symmetric monoidal categories we refer to [17, XI.1]). The original proof of coherence relies on a strictification of the monoidal structure of such a category followed by the standard presentation of symmetric groups by generators and relations (for details see [4, §3.3, Chapter 5]).

Since a coherence result of a class of categories deals with canonical arrows, it is natural to formulate it in terms of a category in this class freely generated by a set of objects. This idea is suggested by Voreadou, [25], and it is fairly used in [4]. There is no explicit referring to this approach in [13]—however, one may reconstruct the instant one step proof of Mac Lane's coherence, suggested by the author, through the symmetric monoidal category freely generated by an infinite set of objects. Also, this provides us a possibility to compare the two families of permutoassociahedra.

Given the set $\omega = \{0, 1, 2, \ldots\}$ of generating objects, what is a symmetric monoidal category **SM** with the strict unit I, freely generated by ω? For the set of objects of this category, there is no doubt—this set contains I, and all the other objects are built out of elements of ω with the help of binary operation denoted here by $\cdot$, which is the object part of the required bifunctor. For example, $(2 \cdot ((4 \cdot 2) \cdot 0))$ is an object of this category and we omit, as usual, the outermost parentheses denoting this object by $2 \cdot ((4 \cdot 2) \cdot 0)$.

The arrows of **SM** are equivalence classes of terms. One can choose here different languages to build these terms, and hence to have different equivalence relations, generated by commutative diagrams, that produce the arrows of **SM**.

Our reconstruction of the approach of [13] is that the *terms* are built inductively in the following manner.

($\mathbf{1}$.1) For an object A, the *identity* $\mathbf{1}_A : A \to A$ is an $\mathbf{1}$-term;

($\mathbf{1}$.2) for f and g, $\mathbf{1}$-terms, $(f \cdot g)$ is an $\mathbf{1}$-term.

(α.1) For objects A, B and C distinct from I, $\alpha_{A,B,C} : A \cdot (B \cdot C) \to (A \cdot B) \cdot C$ and $\alpha^{-1}_{A,B,C} : (A \cdot B) \cdot C \to A \cdot (B \cdot C)$ are α-terms;

(α.2) for f an α-term and g an $\mathbf{1}$-term, $(f \cdot g)$ and $(g \cdot f)$ are α-terms.

(τ.1) For $p, q \in \omega$, $\tau_{p,q} : p \cdot q \to q \cdot p$ is a τ-term;

(τ.2) for f a τ-term and g an $\mathbf{1}$-term, $(f \cdot g)$ and $(g \cdot f)$ are τ-terms.

(t.1) Every $\mathbf{1}$-term, α-term and τ-term is a term;

(t.2) if $f : A \to B$ and $g : B \to C$ are terms, then $g \circ f : A \to C$ is a term.

For example,

$$\alpha^{-1}_{2\cdot 4,1,0} \circ (\alpha_{2,4,1} \cdot \mathbf{1}_0) \circ ((\mathbf{1}_2 \cdot \tau_{1,4}) \cdot \mathbf{1}_0) \circ (\alpha^{-1}_{2,1,4} \cdot \mathbf{1}_0) : ((2 \cdot 1) \cdot 4) \cdot 0 \to (2 \cdot 4) \cdot (1 \cdot 0)$$

is a term.

The identities are neutrals for composition. Moreover, $\mathbf{1}_A \cdot \mathbf{1}_B = \mathbf{1}_{A \cdot B}$ and the following diagrams commute for f, g and $f \cdot (g \cdot h)$ being α-terms or τ-terms.

$$\begin{tikzcd}
A \cdot B \arrow[r, "f \cdot \mathbf{1}"] \arrow[d, "\mathbf{1} \cdot g"'] & A' \cdot B \arrow[d, "\mathbf{1} \cdot g"] \\
A \cdot B' \arrow[r, "f \cdot \mathbf{1}"'] & A' \cdot B'
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
A \cdot (B \cdot C) \arrow[r, "\alpha"] \arrow[d, "\mathbf{1}"'] & (A \cdot B) \cdot C \arrow[d, "\mathbf{1}"] \arrow[dl, "\alpha^{-1}"'] \\
A \cdot (B \cdot C) \arrow[r, "\alpha"'] & (A \cdot B) \cdot C
\end{tikzcd} \tag{2}$$

$$\begin{tikzcd}
 & A \cdot (B \cdot (C \cdot D)) \arrow[dl, "\mathbf{1} \cdot \alpha"'] \arrow[ddr, "\alpha"] & \\
A \cdot ((B \cdot C) \cdot D) \arrow[d, "\alpha"'] & & \\
(A \cdot (B \cdot C)) \cdot D \arrow[dr, "\alpha \cdot \mathbf{1}"'] & & (A \cdot B) \cdot (C \cdot D) \arrow[dl, "\alpha"] \\
 & ((A \cdot B) \cdot C) \cdot D &
\end{tikzcd} \tag{3}$$

$$\begin{tikzcd}
A \cdot (B \cdot C) \arrow[r, "\alpha"] \arrow[d, "f \cdot (g \cdot h)"'] & (A \cdot B) \cdot C \arrow[d, "(f \cdot g) \cdot h"] \\
A' \cdot (B' \cdot C') \arrow[r, "\alpha"'] & (A' \cdot B') \cdot C'
\end{tikzcd} \tag{4}$$

$$\begin{tikzcd}
p \cdot q \arrow[r, "\tau"] \arrow[d, "\mathbf{1}"'] & q \cdot p \arrow[dl, "\tau"] \\
p \cdot q &
\end{tikzcd} \tag{5}$$

$$\begin{tikzcd}[column sep=small]
 & & p \cdot (q \cdot r) \arrow[r, "\mathbf{1} \cdot \tau"] \arrow[dl, "\alpha"'] & p \cdot (r \cdot q) \arrow[dr, "\alpha"] & & \\
 & (p \cdot q) \cdot r \arrow[dl, "\tau \cdot \mathbf{1}"'] & & & (p \cdot r) \cdot q \arrow[dr, "\tau \cdot \mathbf{1}"] & \\
(q \cdot p) \cdot r \arrow[d, "\alpha^{-1}"'] & & & & & (r \cdot p) \cdot q \arrow[d, "\alpha^{-1}"] \\
q \cdot (p \cdot r) \arrow[dr, "\mathbf{1} \cdot \tau"'] & & & & & r \cdot (p \cdot q) \arrow[dl, "\mathbf{1} \cdot \tau"] \\
 & q \cdot (r \cdot p) \arrow[dr, "\alpha"'] & & & r \cdot (q \cdot p) \arrow[dl, "\alpha"] & \\
 & & (q \cdot r) \cdot p \arrow[r, "\tau \cdot \mathbf{1}"'] & (r \cdot q) \cdot p & &
\end{tikzcd} \tag{6}$$

All these equations are assumed to *hold in contexts*. For example, the equation (4) in the context $\cdot D$ reads

$$\begin{tikzcd}
(A \cdot (B \cdot C)) \cdot D \arrow[r, "\alpha \cdot \mathbf{1}"] \arrow[d, "(f \cdot (g \cdot h)) \cdot \mathbf{1}"'] & ((A \cdot B) \cdot C) \cdot D \arrow[d, "((f \cdot g) \cdot h) \cdot \mathbf{1}"] \\
(A' \cdot (B' \cdot C')) \cdot D \arrow[r, "\alpha \cdot \mathbf{1}"'] & ((A' \cdot B') \cdot C') \cdot D
\end{tikzcd}$$

Out of the category **SM**, one can build a (*pseudo*)*graph* $\mathcal{G}$ (in the terminology of [11]) whose *vertices* are the objects of **SM**, while A and B are joined by an *edge* in $\mathcal{G}$, when there is an α-term, or a τ-term $f : A \to B$. Since every such term has an inverse, the edges are undirected. There are no multiple edges. However, this graph contains loops since, for example, there is an edge joining $p \cdot p$ with itself, witnessed by the term $\tau_{p,p}$.

Every connected component of $\mathcal{G}$ contains the objects with the same number of occurrences of elements of ω. Hence, the object I, as well as every element of ω, makes a singleton connected component of **SM**. For the coherence question, we are interested in *diversified* connected components, i.e. the connected components containing the objects of **SM** without repeating the elements of ω. In particular, for every $n \in \omega$, we are interested in the connected component $\mathcal{G}_n$ of $\mathcal{G}$ containing the object of the form

$$0 \cdot (1 \cdot (\ldots \cdot n) \ldots).$$

It is straightforward to see that $\mathcal{G}_n$ does not contain loops.

In order to establish the symmetric monoidal coherence (cf. [4, §3.3,Chapter 5]), it suffices to show that for every $n \in \omega$, every diagram involving the objects of $\mathcal{G}_n$ commutes in **SM**. A topological proof of this fact consists in finding an n-dimensional CW-ball, whose 0-cells are the vertices of $\mathcal{G}_n$, 1-cells are the edges of $\mathcal{G}_n$, and 2-cells correspond to quadrilaterals (1) and (4), pentagons (3) and dodecagons (6) given above. This all is done by Kapranov in [13].

Our approach is a bit different. We construct a category $\mathbf{SM}^*$ with the same objects as **SM**, for which we show that is, up to renaming of arrows, the same as **SM**.

To fix the notation we define the *left*, *right* and *middle contexts* in the following way. The symbol $\Box$ is a left and a right context. If L is a left and R is a right context, while A is an object of $\mathbf{SM}^*$, then $(A \cdot L)$ is a left context, $(R \cdot A)$ is a right context, and $(R \cdot L)$ is a middle context.

If L, R and M are left, right and middle contexts, respectively, and $a, b \in \omega$, then La denotes the object of $\mathbf{SM}^*$ obtained by replacing the occurrence of $\Box$ in L by a, and we define the objects bR and bMa in the same way. It is obvious that for every object A of $\mathbf{SM}^*$ different from I, there exists a unique left context L and a unique $a \in \omega$, namely the rightmost occurrence of an element of ω in A, such that A is La. The same holds for right contexts and if A contains at least one $\cdot$, the same holds for middle contexts.

Our definition of *terms* of $\mathbf{SM}^*$ is as for **SM** save that the clauses (τ.1) and (τ.2) are replaced by the clause

(σ) For objects La and bR, $\sigma_{La,bR} : La \cdot bR \to Lb \cdot aR$ is a σ-term,

and in (t.1), "τ-term" is replaced by "σ-term".

For example,

$$\sigma_{2\cdot 1,4\cdot 0} \circ \alpha^{-1}_{2\cdot 1,4,0} : ((2 \cdot 1) \cdot 4) \cdot 0 \to (2 \cdot 4) \cdot (1 \cdot 0)$$

is a term.

To define the arrows of $\mathbf{SM}^*$, we use the equivalence relation on terms, which is generated by the commutative diagrams (1), (2), (3) and (4) (save that f, g and $f \cdot (g \cdot h)$ are α-terms) and the following commutative diagrams involving σ's.

$$\begin{tikzcd} La \cdot bR \arrow[r, "\sigma"] \arrow[d, "\mathbf{1}"'] & Lb \cdot aR \arrow[dl, "\sigma"] \\ La \cdot bR \end{tikzcd} \tag{7}$$

For the diagram (8), it is assumed that $f : La \to L'a$ and $g : bR \to bR'$ are α-terms and that $f^a_b : Lb \to L'b$ and $g^b_a : aR \to aR'$ are corresponding α-terms obtained by substituting b for the distinguished occurrence of a in f, and vice versa for g.

$$
\begin{tikzcd}
La\cdot bR \arrow[r, "f\cdot\mathbf{1}"] \arrow[d, "\sigma"'] & L'a\cdot bR \arrow[d, "\sigma"] \\
Lb\cdot aR \arrow[r, "f^a_b\cdot\mathbf{1}"'] & L'b\cdot aR
\end{tikzcd}
\qquad
\begin{tikzcd}
La\cdot bR \arrow[r, "\mathbf{1}\cdot g"] \arrow[d, "\sigma"'] & La\cdot bR' \arrow[d, "\sigma"] \\
Lb\cdot aR \arrow[r, "\mathbf{1}\cdot g^b_a"'] & Lb\cdot aR'
\end{tikzcd}
\tag{8}
$$

$$
\begin{tikzcd}[column sep=small]
& La\cdot(bMc\cdot dR) \arrow[r, "\sigma"] \arrow[dl, "\alpha"'] & Lb\cdot(aMc\cdot dR) \arrow[dr, "\alpha"] & \\
(La\cdot bMc)\cdot dR \arrow[d, "\sigma"'] & & & (Lb\cdot aMc)\cdot dR \arrow[d, "\sigma"] \\
(La\cdot bMd)\cdot cR \arrow[dr, "\alpha^{-1}"'] & & & (Lb\cdot aMd)\cdot cR \arrow[dl, "\alpha^{-1}"] \\
& La\cdot(bMd\cdot cR) \arrow[r, "\sigma"'] & Lb\cdot(aMd\cdot cR) &
\end{tikzcd}
\tag{9}
$$

$$
\begin{tikzcd}[column sep=tiny]
& & La\cdot(b\cdot cR) \arrow[r, "\sigma"] \arrow[dl, "\alpha"'] & Lb\cdot(a\cdot cR) \arrow[dr, "\alpha"] & & \\
& (La\cdot b)\cdot cR \arrow[dl, "\sigma"'] & & & (Lb\cdot a)\cdot cR \arrow[dr, "\sigma"] & \\
(La\cdot c)\cdot bR \arrow[d, "\alpha^{-1}"'] & & & & & (Lb\cdot c)\cdot aR \arrow[d, "\alpha^{-1}"] \\
La\cdot(c\cdot bR) \arrow[dr, "\sigma"'] & & & & & Lb\cdot(c\cdot aR) \arrow[dl, "\sigma"] \\
& Lc\cdot(a\cdot bR) \arrow[dr, "\alpha"'] & & & Lc\cdot(b\cdot aR) \arrow[dl, "\alpha"] & \\
& & (Lc\cdot a)\cdot bR \arrow[r, "\sigma"'] & (Lc\cdot b)\cdot aR & &
\end{tikzcd}
\tag{10}
$$

The equations (1), (2), (3) and (4) are assumed to hold in contexts, while the contexts make no sense for the equations (7), (8), (9) and (10).

It is more or less straightforward to define the σ-terms by τ-terms and α-terms, and to show by induction that the equations (7), (8), (9) and (10) hold in **SM** for defined σ's. The equations (5), (4) with $f\cdot(g\cdot h)$ being a τ-term, (1) with f, g being τ-terms and (6), respectively, are essential for this proof. However, if one is not interested in a new topological proof of Mac Lane's coherence, then this coherence result instantly delivers the commutativity of the above diagrams in **SM**.

For the other direction, the τ-terms are easily defined in terms of σ-terms and α-terms. Our family of polytopes (see Section 5) will guarantee that all the equations of **SM** hold in $\mathbf{SM}^*$ for defined τ-terms. Hence, **SM** and $\mathbf{SM}^*$ are, up to renaming of arrows, the same.

Let $\mathcal{G}^*$ be a (pseudo)graph with the same *vertices* as $\mathcal{G}$, while A and B are joined by an *edge* in $\mathcal{G}^*$, when there is an α-term, or a σ-term $f : A \to B$. As in the case of $\mathcal{G}$, we denote by $\mathcal{G}^*_n$ the connected component of $\mathcal{G}^*$ containing the object

$$0\cdot(1\cdot(\ldots\cdot n)\ldots).$$

It is evident that $\mathcal{G}^*_n$ is n-regular, i.e. it has exactly n edges incident to every vertex.

A graph G is a *graph of a polytope* P when the vertices and edges of P are in one-to-one correspondence with the vertices and edges of G, and this correspondence respects the incidence relation. By a result of Blind and Mani-Levitska, [1] (see also Kalai's elegant proof, [12]), one can conclude that if there is an n-dimensional simple polytope whose graph is $\mathcal{G}^*_n$, then all such polytopes are combinatorially equivalent. Our goal is to geometrically realise such a polytope for every n, and to show that its 2-faces correspond to quadrilaterals (1), (4) and (8), pentagons (3), octagons (9) and dodecagons (10).

3. A two-fold nested set construction

A family of simplicial complexes that are closely related to the face lattices of a family of polytopes, which we call simple permutoassociahedra, is introduced in this section. Our construction of this family of simplicial complexes is an example of the two-fold construction of complexes of nested sets for simplicial complexes, which is thoroughly investigated in [19]. Since we are here interested in a single family of complexes, there is no reason to repeat the theory of nested set complexes in its full generality and we present only the facts necessary for our construction. We refer to [19] for a detailed treatment of the theory.

An *abstract simplicial complex* on a set T is a non-empty collection K of finite subsets of T closed under taking inclusions, i.e. if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$. The elements of K are called *simplices.* It is assumed that $\{v\}$ is a simplex of K for every element $v \in T$. An abstract simplicial complex K is *finite* when the number of its simplices is finite, otherwise it is *infinite.* In this contribution we consider only finite abstract simplicial complexes, and we call them shortly *simplicial complexes.*

A simplex $\tau \in K$ is a *face* of a simplex $\sigma \in K$ if $\tau \subseteq \sigma$. A simplex $\sigma \in K$ is called *maximal* when it is not a face of some other simplex from K. If $V_1, \ldots, V_m$ are the maximal simplices of K, than K has a presentation as

$$K = \mathcal{P}(V_1) \cup \ldots \cup \mathcal{P}(V_m).$$

Observe that the maximal simplices of a simplicial complex K are incomparable (with respect to $\subseteq$) finite sets.

Definition 3.1. *A collection $\mathcal{B}$ of non-empty subsets of a finite set V containing all singletons $\{v\}, v \in V$ and satisfying that for any two sets $S_1, S_2 \in \mathcal{B}$ such that $S_1 \cap S_2 \neq \emptyset$, their union $S_1 \cup S_2$ also belongs to $\mathcal{B}$, is called a building set of $\mathcal{P}(V)$.*

Let K be a simplicial complex and let $V_1, \ldots, V_m$ be the maximal simplices of K. A collection $\mathcal{B}$ of some simplices of K is called a building set *of K when for every $i = 1, \ldots, m$, the collection*

$$\mathcal{B}^{V_i} = \mathcal{B} \cap \mathcal{P}(V_i)$$

is a building set of $\mathcal{P}(V_i)$.

The notion of a building set of a finite-meet semilattice introduced in [6, Definition 2.2] reduces to this notion when the semilattice is a simplicial complex (cf. [19, Section 3]).

For a family of sets N, we say that $\{X_1, \ldots, X_m\} \subseteq N$ is an N-*antichain*, when $m \geq 2$ and $X_1, \ldots, X_m$ are mutually incomparable with respect to $\subseteq$.

Definition 3.2. *Let $\mathcal{B}$ be a building set of a simplicial complex K. We say that $N \subseteq \mathcal{B}$ is a* nested set *with respect to $\mathcal{B}$, when the union of every N-antichain is an element of $K - \mathcal{B}$.*

This definition is in accordance with [6, Definition 2.7] (cf. [19, Section 3]).

Now we proceed to the main construction of the paper. We start with the set $X = \{0, \ldots, n\}$, $n \geq 1$, i.e. the ordinal $n + 1$. (The case when $n = 0$ is trivial.) Let C_0 be the simplicial complex $\mathcal{P}(X) - \{X\}$, i.e. the family of subsets of X with at most n elements, and let $\mathcal{B}_0 = C_0 - \{\emptyset\}$. In the literature, C_0 is also known as the *boundary complex* $\partial\Delta^n$ of the abstract n-simplex Δ^n, and obviously $\mathcal{B}_0$ is a building set of the simplicial complex C_0 according to Definition 3.1.

A set $N \subseteq \mathcal{B}_0$ such that the union of every N-antichain belongs to $C_0 - \mathcal{B}_0$ is called 0-*nested*. According to Definition 3.2, every 0-nested set is a nested set with respect to $\mathcal{B}_0$. Since $C_0 - \mathcal{B}_0 = \{\emptyset\}$, every two members of a 0-nested set are comparable.

It is evident that a subset of a 0-nested set is a 0-nested set—thus the family of all 0-nested sets makes a simplicial complex, which we denote by C_1. The maximal 0-nested sets are of the form

$$\big\{\{i_n, \ldots, i_1\}, \ldots, \{i_n, i_{n-1}\}, \{i_n\}\big\},$$

where $i_1, \ldots, i_n$ are mutually distinct elements of X. Such a 0-nested set corresponds to the permutation

$$i_0 i_1 \ldots i_n$$

of X, where $\{i_0\} = X - \{i_1, \ldots, i_n\}$. For example, when $X = \{0, 1, 2, 3\}$,

$$\big\{\{1, 0, 3\}, \{1, 0\}, \{1\}\big\}$$

is a maximal 0-nested set that corresponds to the permutation 2301 (see the picture at the end of Section 5).

We say that a polytope P (*geometrically*) *realises* a simplicial complex K, when the semilattice obtained by removing the bottom (the empty set) from the face lattice of P is isomorphic to $(K, \supseteq)$. The n-dimensional permutohedron (cf. [18, Definition 4.1]) realises the simplicial complex C_1. To prove this, one has to rely on [19, Proposition 3.4] and well known facts about nested set complexes related to permutohedra (cf. [7], [2], [20] and [5]).

In this one-to-one correspondence, the vertices of the permutohedron correspond to the maximal 0-nested sets. Hence, there are $(n+1)!$ maximal 0-nested sets. Also, the facets of the permutohedron correspond to the singleton nested sets of the form $\{A\}$, where $A \in \mathcal{B}_0$. It is more intuitive to denote the facets of this permutohedron by ordered partitions of X with two blocks. With singleton nested sets we just economise a little bit—our nested set $\{A\}$ corresponds to the ordered partition whose first block is $X - A$ and the second is A.

For a maximal 0-nested set

$$\big\{\{i_n, \ldots, i_1\}, \ldots, \{i_n, i_{n-1}\}, \{i_n\}\big\},$$

consider the following *path graph* with n vertices and $n-1$ edges:

$$\{i_n, \ldots, i_1\} - \ldots - \{i_n, i_{n-1}\} - \{i_n\}.$$

We say that a set of vertices of this graph is *connected*, when this is the set of vertices of a connected subgraph of this graph.

Let $\mathcal{B}_1 \subseteq C_1$ be the family of all sets of the form

$$\big\{\{i_{k+l}, \ldots, i_k, \ldots, i_1\}, \ldots, \{i_{k+l}, \ldots, i_k, i_{k-1}\}, \{i_{k+l}, \ldots, i_k\}\big\},$$

where $1 \leq k \leq k+l \leq n$ and $i_1, \ldots, i_{k+l}$ are mutually distinct elements of X, i.e. $\mathcal{B}_1$ is the set of all non-empty connected sets of vertices of the path graphs that correspond to all maximal 0-nested sets. According to Definition 3.1, we have that $\mathcal{B}_1$ is a building set of the simplicial complex C_1.

Remark 3.1. *Every member of $C_1 - \mathcal{B}_1$ is of the form $\{A_1, \ldots, A_r\}$, where*

$$2 \leq r \leq n-1 \;\&\; X \supset A_1 \supset \ldots \supset A_r \;\&\; \exists i\, |A_i - A_{i+1}| \geq 2.$$

Example 3.2. *For $X = \{0, 1, 2\}$, we have that $\mathcal{B}_1$ is*

$$\Big\{\big\{\{0\}\big\},\ \big\{\{1\}\big\},\ \big\{\{2\}\big\},\ \big\{\{0,1\}\big\},\ \big\{\{0,2\}\big\},\ \big\{\{1,2\}\big\},$$
$$\big\{\{0,1\},\{0\}\big\},\ \big\{\{0,2\},\{0\}\big\},\ \big\{\{0,1\},\{1\}\big\},$$
$$\big\{\{1,2\},\{1\}\big\},\ \big\{\{0,2\},\{2\}\big\},\ \big\{\{1,2\},\{2\}\big\}\Big\}.$$

A set $N \subseteq \mathcal{B}_1$ is 1-*nested* when the union of every N-antichain belongs to $C_1 - \mathcal{B}_1$. According to Definition 3.2, every 1-nested set is a nested set with respect to $\mathcal{B}_1$. Again, it is evident that the family of all 1-nested sets makes a simplicial complex, which we denote by C. As a corollary of the following proposition, we have that C is a flag complex.

Proposition 3.3. *A set $N \subseteq \mathcal{B}_1$ is 1-nested when the union of every two incomparable elements of N belongs to $C_1 - \mathcal{B}_1$.*

Proof. Let $\mathcal{A}$ be an N-antichain. By our assumption, every two different members of $\mathcal{A}$ are of the form

$$\{A_{i1}, \ldots, A_{ir}\} \quad \text{and} \quad \{A_{j1}, \ldots, A_{jq}\},$$

with $A_{i1} \supset \ldots \supset A_{ir} \supset A_{j1} \supset \ldots \supset A_{jq}$, and

$$|A_{ik} - A_{i(k+1)}| = 1 = |A_{jk} - A_{j(k+1)}|, \quad |A_{ir} - A_{j1}| \geq 2.$$

Hence $\mathcal{A}$ can be linearly ordered in a natural way, and we may conclude that $\bigcup \mathcal{A}$ is a subset of a maximal 0-nested set, which means that it is in C_1. The "gaps" arose by provisos $|A_{ir} - A_{j1}| \geq 2$ guarantee that $\bigcup \mathcal{A}$ is not in $\mathcal{B}_1$. $\square$

Example 3.4. *For $X = \{0, 1, 2\}$, we have the following 12 maximal 1-nested sets:*

$$\Big\{\{\{1,2\},\{2\}\},\{\{1,2\}\}\Big\}, \quad \Big\{\{\{1,2\},\{2\}\},\{\{2\}\}\Big\}, \quad \Big\{\{\{0,2\},\{2\}\},\{\{2\}\}\Big\},$$
$$\Big\{\{\{0,2\},\{2\}\},\{\{0,2\}\}\Big\}, \quad \Big\{\{\{0,2\},\{0\}\},\{\{0,2\}\}\Big\}, \quad \Big\{\{\{0,2\},\{0\}\},\{\{0\}\}\Big\},$$
$$\Big\{\{\{0,1\},\{0\}\},\{\{0\}\}\Big\}, \quad \Big\{\{\{0,1\},\{0\}\},\{\{0,1\}\}\Big\}, \quad \Big\{\{\{0,1\},\{1\}\},\{\{0,1\}\}\Big\},$$
$$\Big\{\{\{0,1\},\{1\}\},\{\{1\}\}\Big\}, \quad \Big\{\{\{1,2\},\{1\}\},\{\{1\}\}\Big\}, \quad \Big\{\{\{1,2\},\{1\}\},\{\{1,2\}\}\Big\},$$

and C is the union of power sets of these sets.

Example 3.5. *If $X = \{0, 1, 2, 3\}$ and $M = \{\{1, 0, 3\}, \{1, 0\}, \{1\}\}$, then*

$$N = \Big\{M, \{\{1\}\}, \{\{1,0,3\}\}\Big\}, \quad \Big\{M, \{\{1,0,3\},\{1,0\}\}, \{\{1,0,3\}\}\Big\},$$

$$\Big\{M, \{\{1,0\},\{1\}\}, \{\{1\}\}\Big\}, \quad \Big\{M, \{\{1,0,3\},\{1,0\}\}, \{\{1,0\}\}\Big\},$$

$$\text{and} \quad \Big\{M, \{\{1,0\},\{1\}\}, \{\{1,0\}\}\Big\},$$

are 5 out of 120 maximal 1-nested sets.

In Example 3.5, every maximal 1-nested set, except N, is such that all of its members are comparable (it contains no antichains), while N contains the N-antichain $\{\{\{1\}\}, \{\{1, 0, 3\}\}\}$ whose union is $\{\{1\}, \{1, 0, 3\}\} \notin \mathcal{B}_1$.

If N is a 1-nested set, then there is a maximal 0-nested set M such that every member of N is a subset of M, i.e. $N \subseteq \mathcal{B}_1^M = \mathcal{B}_1 \cap \mathcal{P}(M)$. In this case, we say that N is *derived* from M. In Example 3.5, the five displayed 1-nested sets are derived from M. If N is a maximal 1-nested set, then there is unique such M, namely the maximal 0-nested set contained in N. This also holds for every 1-nested set that contains a maximal 0-nested set. Otherwise, if N does not contain a maximal 0-nested set, then there is not a unique such M.

Remark 3.6. *Every maximal 0 and 1-nested set is of cardinality n.*

Remark 3.7. *Every element of $\mathcal{B}_1$ is contained in a maximal 1-nested set.*

For a maximal 0-nested set M, let C^M be the *link* of M in C, i.e. C^M is the set of subsets not containing M of the elements of C that contain M. This simplicial complex is related to the $(n-1)$-dimensional associahedron (Stasheff polytope, see [23] and [24]) in the same way as the simplicial complex C_1 has been related to the n-dimensional permutohedron.

The vertices of the $(n-1)$-dimensional associahedron correspond to all the

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

different ways of bracketing a string of $n+1$ letters. The number c_n is the nth *Catalan* number. Two vertices are adjacent if they correspond to a single application of the associative law. In general, k-faces of the associahedron are in bijection with the set of correct bracketings of a string of $n+1$ letters with $n-k$ pairs of brackets. Two vertices lie in the same k-face if and only if the corresponding complete bracketings could be reduced, by removing k pairs of brackets, to the same bracketing of the string of $n+1$ letters with $n-k$ pairs of brackets. Figure 1 depicts the $(n-1)$-dimensional associahedra for $n = 2, 3$ and 4.

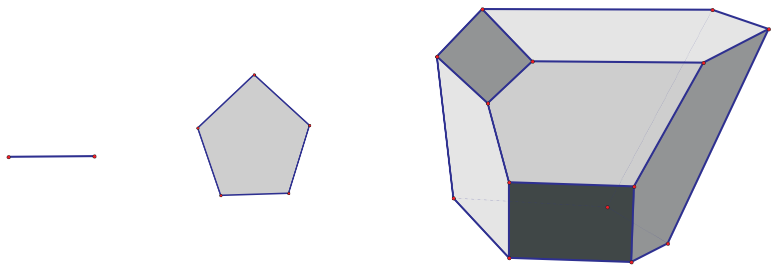


FIGURE 1. K_3, K_4 and K_5

The $(n-1)$-dimensional associahedron realises C^M. This is again proved by relying on well known facts about nested set complexes related to associahedra (cf. [7], [2], [20] and [5]). Hence, there are c_n maximal 1-nested sets derived from a maximal 0-nested set. From this, one concludes that there are all in all

$$\frac{(2n)!}{n!}$$

maximal 1-nested sets. A correspondence between the maximal 1-nested sets and complete bracketings of permuted products of the elements of $\{0, \ldots, n\}$ is given in Section 4.

The main question related to the simplicial complex C is the following.

Question 3.8. *Is there an n-dimensional realisation of C?*

The positive answer to this question is given in Section 5. But first, in Section 4, we show that if such a polytope exists, then its graph is $\mathcal{G}_n^*$ and the 2-faces of this polytope correspond to the diagrams (1), (4), (8), (3), (9) and (10).

4. THE 0, 1 AND 2-FACES OF C

In order to establish the correspondence announced above, we investigate the elements of the simplicial complex C that should correspond to the vertices, edges and two dimensional faces of a polytope, which provides the positive answer to Question 3.8. Respectively, we call these elements of C the 0-faces, 1-faces and 2-faces.

4.0. **0-faces.** The 0-*faces* of C are the maximal 1-nested sets. Every 0-face corresponds to a unique complete bracketing of a permuted product of the elements of $\{0,\ldots,n\}$ in the following way. Let V be a 0-face derived from

$$M = \big\{\{i_n,\ldots,i_1\},\ldots,\{i_n,i_{n-1}\},\{i_n\}\big\}.$$

We start with $n+1$ place holders separated by n dots, which are labelled by the elements of M in the descending (with respect to $\subseteq$) order:

$$\_\_ \overset{\{i_n,\ldots,i_1\}}{\cdot} \_\_ \cdots \_\_ \overset{\{i_n,i_{n-1}\}}{\cdot} \_\_ \overset{\{i_n\}}{\cdot} \_\_.$$

Each label denotes the elements of X to the right of the labelled dot. Hence we have the product:

$$i_0 \overset{\{i_n,\ldots,i_1\}}{\cdot} i_1 \overset{\{i_n,\ldots,i_2\}}{\cdot} \cdots \overset{\{i_n,i_{n-1}\}}{\cdot} i_{n-1} \overset{\{i_n\}}{\cdot} i_n,$$

where $\{i_0\} = X - \{i_1,\ldots,i_n\}$. Eventually, for every element A of V, one has to insert the pair of brackets that include the labels contained in A.

Example 4.1. *For* $X=\{0,1,2,3\}$, $M=\{\{1,0,3\},\{1,0\},\{1\}\}$, *and* $V=\{M,\{\{1\}\},\{\{1,0,3\}\}\}$, *we have that* V *corresponds to the following complete bracketing*

$$((2 \overset{\{1,0,3\}}{\cdot} 3) \overset{\{1,0\}}{\cdot} (0 \overset{\{1\}}{\cdot} 1)).$$

In the picture given at the end of Section 5, the 0-face V corresponds to the vertex incident to the pentagon covering the vertex labelled by 2301, dodecagon covering the label $\{1\}$ and dodecagon covering the label $\{0,1,3\}$.

Also, for every complete bracketing P of a product $i_0 \cdot i_1 \cdot \ldots \cdot i_n$, one can find a unique 0-face of C that corresponds to P. Hence, the 0-faces of C are in one-to-one correspondence with the vertices of $\mathcal{G}^*_n$.

Remark 4.2. *Since the face lattice of a simple polytope is completely determined by the bipartite graph of incidence relation between the vertex set and the set of facets, we have the following alternative way to define the lattice structure, which may replace our simplicial complex C. This is suggested to us by G.M. Ziegler.*

As the vertex set take all the complete bracketings of permuted products of $X=\{0,\ldots,n\}$, and as the set of facets take all the ordered partitions of X into k blocks, $k\geq 2$, where only the first and the last block are allowed to contain more than one element. A vertex V, whose underlying permutation is $i_0i_1\ldots i_n$, and a facet F are incident when there is a pair of corresponding brackets in V with i_j being the leftmost and i_k being the rightmost element in the scope of these brackets, such that F is the ordered partition of X whose first block is $\{i_0,\ldots,i_j\}$ followed by $k-j-1$ singleton blocks $\{i_{j+1}\},\ldots,\{i_{k-1}\}$, and the last block is $\{i_k,\ldots,i_n\}$.

If we assign to an element of $\mathcal{B}_1$ of the form

$$\big\{\{i_{k+l},\ldots,i_k,\ldots,i_1\},\ldots,\{i_{k+l},\ldots,i_k,i_{k-1}\},\{i_{k+l},\ldots,i_k\}\big\},$$

the ordered partition of X whose first block is $X-\{i_{k+l},\ldots,i_k,\ldots,i_1\}$, the second block is $\{i_1\}$, and so on up to the penultimate block, which is $\{i_{k-1}\}$, while the last block is $\{i_{k+l},\ldots,i_k\}$, then the above analysis of 0-faces explains the incidence relation given in this remark.

4.1. **1-faces.** The 1-*faces* of C are obtained from the 0-faces by removing a single element. There are two main types of 1-faces of C; a 1-face of the *first type* contains a maximal 0-nested set, and a 1-face of the *second type* is obtained from a 0-face by removing its maximal 0-nested set.

If E is a 1-face of the first type, and

$$M = \{\{i_n,\ldots,i_1\},\ldots,\{i_n,i_{n-1}\},\{i_n\}\}$$

is the maximal 0-nested set contained in E, then a unique bracketing of the product $i_0 \cdot i_1 \cdot \ldots \cdot i_n$, with one pair of brackets omitted, is assigned to E in the same way as a complete bracketing has been assigned to a 0-face. There are exactly two 0-faces that include a 1-face E of the first type. They correspond to two possible insertions of pairs of brackets in the bracketing corresponding to E. Hence, every 1-face of the first type corresponds to an α-term.

Example 4.3. *For* $X = \{0,1,2,3\}$, $M = \{\{1,0,3\},\{1,0\},\{1\}\}$, *and* $E = \{M, \{\{1\}\}\}$, *we have that* E *corresponds to the following bracketing*

$$(2 \overset{\{1,0,3\}}{\cdot} 3 \overset{\{1,0\}}{\cdot} (0 \overset{\{1\}}{\cdot} 1)).$$

The 0-face V *from Example 4.1 and the 0-face* $V' = \{M, \{\{1,0\},\{1\}\}, \{\{1\}\}\}$, *which corresponds to the complete bracketing*

$$(2 \cdot (3 \cdot (0 \cdot 1))),$$

are the 0-faces that include E. *In the picture given at the end of Section 5, the 1-face* E *corresponds to the edge incident to the pentagon covering the vertex labelled by 2301 and dodecagon covering the label* $\{1\}$.

As before, for every bracketing P of a product $i_0 \cdot i_1 \cdot \ldots \cdot i_n$, with one pair of brackets omitted, one can find a unique 1-face of the first type that corresponds to P.

In order to describe the 1-faces of the second type, we define some notions, whose more general form is introduced in [5]. A *construction* of a path graph

$$v_1 - v_2 - \ldots - v_n$$

$n \geq 1$, is a subset of $\mathcal{P}(\{v_1, \ldots, v_n\})$ inductively defined as follows:

(1) if $n = 1$, then $\{\{v_1\}\}$ is the construction of this graph with a single vertex v_1;

(2) if $n > 1$, $1 \leq j \leq n$ and L, R are constructions of the path graphs

$$v_1 - \ldots - v_{j-1} \quad \text{and} \quad v_{j+1} - \ldots - v_n$$

(if $j = 1$, then $L = \emptyset$, and if $j = n$, then $R = \emptyset$), then

$$\big\{\{v_1, \ldots, v_n\}\big\} \cup L \cup R$$

is a construction of the path graph

$$v_1 - v_2 - \ldots - v_n.$$

Remark 4.4. *If* K *is a construction of* $v_1 - \ldots - v_n$, *then* $\{v_1, \ldots, v_n\} \in K$.

Remark 4.5. *Every construction of* $v_1 - \ldots - v_n$ *has exactly* n *elements.*

The following proposition stems from [5, Proposition 6.11].

Proposition 4.6. V *is a 0-face derived from*

$$\big\{\{i_n, \ldots, i_1\}, \ldots, \{i_n, i_{n-1}\}, \{i_n\}\big\}$$

iff V *is a construction of the path graph*

$$\{i_n, \ldots, i_1\} - \ldots - \{i_n, i_{n-1}\} - \{i_n\}.$$

From this proposition and the relation between the simplicial complex C^M and the $(n-1)$-dimensional associahedron mentioned in Section 3, one can extract the following recurrence relation for the Catalan numbers

$$c_0 = 1 \quad \text{and} \quad c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}.$$

For $K \in C$ and $A \in K$, we say that $a \in A$ is A-*superficial* with respect to K, when for every $B \in K$ such that $B \subset A$, we have that $a \notin B$. The number of A-superficial elements with respect to K is denoted by $\nu_K(A)$. From the definition of construction and Proposition 4.6 we have the following.

Remark 4.7. *If K is a 0-face, then for every $A \in K$ we have that $\nu_K(A) = 1$.*

Let E be a 1-face of the second type, i.e. E is obtained from a 0-face V by removing its maximal 0-nested set M. By Remark 4.7, $\nu_V(M) = 1$, and hence, $\bigcup E$ is of the form

$$\big\{\{i_n,\dots,i_1\},\dots,\{i_n,\dots,i_{j-1}\},\{i_n,\dots,i_{j+1}\},\dots,\{i_n\}\big\},$$

which in the limit cases reads

$$\big\{\{i_n,\dots,i_2\},\dots,\{i_n\}\big\} \quad \text{or} \quad \big\{\{i_n,\dots,i_1\},\dots,\{i_n,i_{n-1}\}\big\}.$$

By repeating the procedure we used for 0-faces and 1-faces of the first type, starting with $\bigcup E$, the following product (with two place holders vacant) is obtained

$$i_0 \overset{\{i_n,\dots,i_1\}}{\cdot} i_1 \cdot \ldots \cdot i_{j-2} \overset{\{i_n,\dots,i_{j-1}\}}{\cdot} \_\_ \cdot \_\_ \overset{\{i_n,\dots,i_{j+1}\}}{\cdot} i_{j+1} \cdot \ldots \overset{\{i_n\}}{\cdot} i_n,$$

and a complete bracketing of this product is reconstructed in the same way as for 0-faces. (Formally, the outermost brackets are omitted, but we take them for granted.) Note that this product must contain the following brackets:

$$(i_0 \overset{\{i_n,\dots,i_1\}}{\cdot} i_1 \cdot \ldots \cdot i_{j-2} \overset{\{i_n,\dots,i_{j-1}\}}{\cdot} \_\_) \cdot (\_\_ \overset{\{i_n,\dots,i_{j+1}\}}{\cdot} i_{j+1} \cdot \ldots \overset{\{i_n\}}{\cdot} i_n).$$

There are exactly two 0-faces that include a 1-face E of the second type. They correspond to two possible permutations of i_{j-1}, i_j in the vacant positions of the bracketing corresponding to E. Hence, every 1-face of the second type corresponds to a σ-term.

Example 4.8. *For $X = \{0,1,2,3\}$, and $E = \{\{\{1\}\},\{\{1,0,3\}\}\}$, we have that E corresponds to the following bracketing*

$$(2 \overset{\{1,0,3\}}{\cdot} \_\_) \cdot (\_\_ \overset{\{1\}}{\cdot} 1).$$

The 0-face V from Example 4.1 and the 0-face

$$V'' = \{\{\{1,3,0\},\{1,3\},\{1\}\},\{\{1\}\},\{\{1,3,0\}\}\},$$

which corresponds to the complete bracketing

$$((2 \cdot 0) \cdot (3 \cdot 1)),$$

are the 0-faces that include E. In the picture given at the end of Section 5, the 1-face E corresponds to the edge incident to the dodecagon covering the label $\{1\}$ and dodecagon covering the label $\{0,1,3\}$.

As before, for every complete bracketing P of a product with two vacant positions, one being the rightmost in the left factor of P, and the other being the leftmost in the right factor of P, one can find a unique 1-face of the second type that corresponds to P. This, together with the analogous fact concerning the 1-faces of the first type, entails that the 1-faces of C are in one-to-one correspondence with the edges of $\mathcal{G}_n^*$.

4.2. **2-faces.** The 2-*faces* of C are obtained from the 1-faces by removing a single element. There are two main types of 2-faces of C; a 2-face of the *first type* contains a maximal 0-nested set, and a 2-face of the *second type* does not contain a maximal 0-nested set.

If F is a 2-face of the first type, and

$$M = \big\{\{i_n, \dots, i_1\}, \dots, \{i_n, i_{n-1}\}, \{i_n\}\big\}$$

is the maximal 0-nested set contained in F, then a unique bracketing of the product $i_0 \cdot i_1 \cdot \ldots \cdot i_n$, with two pairs of brackets omitted, is assigned to F in the same way as a complete bracketing has been assigned to a 0-face. There are three possible situations:

(1.1) there is $A \in F$ such that $\nu_F(A) = 3$;

(1.2) there are incomparable $A, B \in F$ such that $\nu_F(A) = \nu_F(B) = 2$;

(1.3) there are $A, B \in F$ such that $B \subset A$ and $\nu_F(A) = \nu_F(B) = 2$.

By a straightforward analysis, in the case (1.1), one concludes that F is included in exactly five 1-faces and five 0-faces and it corresponds to Mac Lane's pentagon (3). In the cases (1.2) and (1.3), F is included in exactly four 1-faces (corresponding to α-terms) and four 0-faces. In the case (1.2), F corresponds to a functoriality quadrilateral (1), while in the case (1.3) it corresponds to a naturality quadrilateral (4).

Example 4.9. *Let $X = \{0, 1, 2, 3, 4, 5\}$ and let M be the maximal 0-nested set*

$$\big\{\{0, 1, 2, 3, 4\}, \{0, 1, 2, 3\}, \{0, 1, 2\}, \{0, 1\}, \{0\}\big\}.$$

The 2-face $F_1 = \{M, \{\{0, 1, 2, 3\}, \{0, 1, 2\}, \{0, 1\}, \{0\}\}, \{\{0\}\}\}$, corresponds to the bracketing $(5 \cdot (4 \cdot 3 \cdot 2 \cdot (1 \cdot 0)))$.

$$\begin{tikzcd}
 & 5\cdot(4\cdot(3\cdot(2\cdot(1\cdot 0)))) \arrow[dl, dash, "5\cdot(4\cdot(3\cdot 2\cdot(1\cdot 0)))"'] \arrow[ddr, dash, "5\cdot(4\cdot 3\cdot(2\cdot(1\cdot 0)))"] & \\
5\cdot(4\cdot((3\cdot 2)\cdot(1\cdot 0))) \arrow[d, dash, "5\cdot(4\cdot(3\cdot 2)\cdot(1\cdot 0))"'] & & \\
5\cdot((4\cdot(3\cdot 2))\cdot(1\cdot 0)) \arrow[dr, dash, "5\cdot((4\cdot 3\cdot 2)\cdot(1\cdot 0))"'] & & 5\cdot((4\cdot 3)\cdot(2\cdot(1\cdot 0))) \\
 & 5\cdot(((4\cdot 3)\cdot 2)\cdot(1\cdot 0)) \arrow[ur, dash, "5\cdot((4\cdot 3)\cdot 2\cdot(1\cdot 0))"'] &
\end{tikzcd}$$

The 2-face $F_2 = \{M, \{\{0, 1, 2, 3, 4\}, \{0, 1, 2, 3\}\}, \{\{0, 1\}, \{0\}\}\}$, corresponds to the bracketing $((5{\cdot}4{\cdot}3){\cdot}(2{\cdot}1{\cdot}0))$ *and it is easy to make the corresponding functoriality quadrilateral.*

The 2-face $F_3 = \{M, \{\{0, 1, 2, 3\}, \{0, 1, 2\}, \{0, 1\}, \{0\}\}, \{\{0, 1\}, \{0\}\}\}$ corresponds to the bracketing $(5{\cdot}(4{\cdot}3{\cdot}(2{\cdot}1{\cdot}0)))$ *and it is easy to make the corresponding naturality quadrilateral.*

If F is a 2-face of the second type, then there are two possibilities

$$(2.1)\ \ |\bigcup F| = n - 1 \quad \text{or} \quad (2.2)\ \ |\bigcup F| = n - 2.$$

In the case (2.1), a quadrilateral of a form of the diagrams (8) corresponds to F. In the case (2.2), $\bigcup F$ is of the form:

$$\big\{\{i_n, \dots, i_1\}, \dots, \{i_n, i_{n-1}\}, \{i_n\}\big\} - \big\{\{i_n, \dots, i_j\}, \{i_n, \dots, i_k\}\big\},$$

for $1 \leq j < k \leq n$, and we distinguish the following situations:

(2.2.1) if $k-j>1$, then an octagon of a form of the diagram (9) corresponds to F;

(2.2.2) if $k-j=1$, then a dodecagon of a form of the diagram (10) corresponds to F.

Example 4.10. *For $X=\{0,1,2,3,4,5\}$, the 2-face*

$$F_4=\Big\{\{\{0,1,2,3,4\}\},\ \{\{0,1,2\},\{0,1\},\{0\}\},\ \{\{0\}\}\Big\}$$

describes the bracketing $(5\cdot \_\_)\cdot(\_\_\cdot 2\cdot(1\cdot 0))$.

$$\begin{array}{ccc} (5\cdot 4)\cdot(3\cdot(2\cdot(1\cdot 0))) & \xrightarrow{(5\cdot 4)\cdot(3\cdot 2\cdot(1\cdot 0))} & (5\cdot 4)\cdot((3\cdot 2)\cdot(1\cdot 0)) \\ {\scriptstyle (5\cdot \_)\cdot(\_\cdot(2\cdot(1\cdot 0)))}\Big| & & \Big|{\scriptstyle (5\cdot \_)\cdot((\_\cdot 2)\cdot(1\cdot 0))} \\ (5\cdot 3)\cdot(4\cdot(2\cdot(1\cdot 0))) & \xrightarrow[(5\cdot 3)\cdot(4\cdot 2\cdot(1\cdot 0))]{} & (5\cdot 3)\cdot((4\cdot 2)\cdot(1\cdot 0)) \end{array}$$

The 2-face $F_5=\{\{\{0,1,2,3,4\}\},\{\{0,1,2\}\},\{\{0\}\}\}$, describes the bracketing

$$(5\cdot \_\_)\cdot(\_\_\cdot\_\_)\cdot(\_\_\cdot 0),$$

where the first two vacant positions are reserved for 3 and 4, while the last two are reserved for 1 and 2. This face corresponds to an octagon of a form of the diagram (9).

The 2-face $F_6=\{\{\{0,1,2,3,4\}\},\{\{0,1\},\{0\}\},\{\{0\}\}\}$ describes the bracketing

$$(5\cdot \_\_)\cdot\_\_\cdot(\_\_\cdot(1\cdot 0))$$

and it corresponds to a dodecagon of a form of the diagram (10).

5. Simple permutoassociahedra

In this section we present a polytope $\mathbf{PA}_n$, the n-dimensional *simple permutoassociahedron*, and give the positive answer to Question 3.8. We start with the following notation. For $1\le k\le k+l\le n$, let

$$\kappa(k,l)=\frac{3^{k+l+1}-3^{l+1}}{2}+\frac{3^k-3k}{3^n-n-1}.$$

For an element

$$\beta=\big\{\{i_{k+l},\ldots,i_k,\ldots,i_1\},\ldots,\{i_{k+l},\ldots,i_k,i_{k-1}\},\{i_{k+l},\ldots,i_k\}\big\},$$

of $\mathcal{B}_1$, let $\beta^=$ be the equation (*hyperplane* in $\mathbf{R}^{n+1}$)

$$x_{i_1}+2x_{i_2}+\ldots+k(x_{i_k}+\ldots+x_{i_{k+l}})=\kappa(k,l),$$

and let the *halfspace* $\beta^\geq$ and the *open halfspace* $\beta^>$ be defined as $\beta^=$, save that "$=$" is replaced by "$\geq$" and "$>$", respectively.

Remark 5.1. *We have that $\kappa(k,l)=3^{l+k}+\ldots+3^{l+1}+\varepsilon(k)$, for*

$$\varepsilon(k)=\frac{3^k-3k}{3^n-n-1}<1.$$

For π being the hyperplane $x_0+\ldots+x_n=3^{n+1}$ in $\mathbf{R}^{n+1}$, let

$$\mathbf{PA}_n=(\bigcap\{\beta^\geq\mid\beta\in\mathcal{B}_1\})\cap\pi.$$

The rest of this section is devoted to a proof of the following result.

Theorem 5.2. $\mathbf{PA}_n$ *is a simple n-dimensional polytope that realises C.*

For every $0 \leq i \leq n$, we have that $\{\{i\}\} \in \mathcal{B}_1$. Hence, $\mathbf{PA}_n$ is a subset of the n-simplex

$$x_0 + \ldots + x_n = 3^{n+1}, \qquad x_i \geq 3, \quad 0 \leq i \leq n,$$

and therefore a polytope, and not just a polyhedron.

For $\mathcal{B}_1^M$ being defined as $\mathcal{B}_1 \cap \mathcal{P}(M)$, we have the following.

Lemma 5.3. *If $\beta, \gamma \in \mathcal{B}_1$ are such that $\beta^= \cap \gamma^= \cap \mathbf{PA}_n \neq \emptyset$, then there is a maximal 0-nested set M such that $\beta, \gamma \in \mathcal{B}_1^M$.*

Proof. Suppose that

($*$) there is no maximal 0-nested set M such that $\beta, \gamma \in \mathcal{B}_1^M$.

Let $\beta^=$ be the hyperplane

$$y_1 + 2y_2 + \ldots + k(y_k + \ldots + y_{k+l}) = \kappa(k, l),$$

and let $\gamma^=$ be the hyperplane

$$z_1 + 2z_2 + \ldots + m(z_m + \ldots + z_{m+p}) = \kappa(m, p),$$

where $Y = \{y_1, \ldots, y_{k+l}\}$ and $Z = \{z_1, \ldots, z_{m+p}\}$ are subsets of the set of variables $\{x_0, \ldots, x_n\}$. Let Σ be the following sum:

$$y_1 + 2y_2 + \ldots + k(y_k + \ldots + y_{k+l}) + z_1 + 2z_2 + \ldots + m(z_m + \ldots + z_{m+p}).$$

We show that $\Sigma > \kappa(k, l) + \kappa(m, p)$, for every point $a(x_0, x_1, \ldots, x_n) \in \mathbf{PA}_n$, which entails that $\mathbf{PA}_n$ and the hyperplane $\Sigma = \kappa(k, l) + \kappa(m, p)$ are disjoint. Since $\beta^= \cap \gamma^=$ is included in this hyperplane, we conclude that $a \notin \beta^= \cap \gamma^=$. In order to prove this, we use the following fact several times: if $i_1, \ldots, i_m$ are mutually distinct elements of $\{0, \ldots, n\}$, then $\{\{i_1, \ldots, i_m\}\} \in \mathcal{B}_1$, and for $a(x_0, x_1, \ldots, x_n) \in \mathbf{PA}_n$, we have that $x_{i_1} + \ldots + x_{i_m} \geq 3^m$.

(1) If Y and Z are incomparable and $Y \cup Z = \{u_1, \ldots, u_q\}$, then we have that $q > k + l, m + p$ and

$$\Sigma \geq u_1 + \ldots + u_q \geq 3^q \geq \frac{3^{k+l+1} + 3^{m+p+1}}{2} > \kappa(k, l) + \kappa(m, p).$$

(2) If $Z \subset Y$, then, by ($*$), we conclude that either

(2.1) $z_1 = y_i$ for some $i \leq k - 2$ and $\{z_2, \ldots, z_{m+p}\} \subseteq \{y_k, \ldots, y_{k+l}\}$, or

(2.2) $z_1 = y_{k-1}$ and $\{z_2, \ldots, z_{m+p}\} \subset \{y_k, \ldots, y_{k+l}\}$, or

(2.3) $\{z_2, \ldots, z_{m+p}\} \not\subseteq \{y_k, \ldots, y_{k+l}\}$.

In the case (2.1), we have that $k + l - i \geq p + m$, and

$$\begin{aligned}
\Sigma &\geq (y_1 + \ldots + y_{l+k}) + \ldots + (y_{i-1} + \ldots + y_{l+k}) + 2(y_i + \ldots + y_{l+k}) \\
&\geq 3^{l+k} + \ldots + 3^{l+k-i+2} + 2 \cdot 3^{l+k-i+1} \\
&> 3^{l+k} + \ldots + 3^{l+k-i+2} + 3^{l+k-i+1} + 2(3^{l+k-i} + \ldots + 1) \\
&\geq 3^{l+k} + \ldots + 3^{l+1} + 1 + 3^{p+m} + \ldots + 3^{p+1} + 1, \text{ since } k + l - i \geq p + m \\
&> \kappa(k, l) + \kappa(m, p).
\end{aligned}$$

In the case (2.2), we have that $l + 1 \geq p + m$, and we repeat the above calculation with i replaced by $k - 1$.

In the case (2.3), let i be the smallest element of $\{1, \ldots, k - 1\}$ such that $y_i = z_j$ for some $j \in \{2, \ldots, m + p\}$. We have that $k + l - i + 1 \geq p + m$ and

$$\begin{aligned}
\Sigma &\geq (y_1 + \ldots + y_{l+k}) + \ldots + (y_{i-1} + \ldots + y_{l+k}) + 3(y_i + \ldots + y_{l+k}) \\
&\geq 3^{l+k} + \ldots + 3^{l+k-i+2} + 3 \cdot 3^{l+k-i+1} \\
&> 3^{l+k} + \ldots + 3^{l+k-i+2} + 2(3^{l+k-i+1} + \ldots + 1) \\
&\geq 3^{l+k} + \ldots + 3^{l+1} + 1 + 3^{p+m} + \ldots + 3^{p+1} + 1, \text{ since } k+l-i+1 \geq p+m \\
&> \kappa(k,l) + \kappa(m,p).
\end{aligned}$$

(3) If $Z = Y$, then, by $(*)$, there is $i \in \{1, \ldots, k-1\}$ such that for some $r \geq 1$, $\gamma^{=}$ is of the form

$$y_1 + \ldots + (i-1)y_{i-1} + (i+r)y_i + \ldots = \kappa(m,p),$$

where the coefficient of every y_j, $j > i$, hidden in "$\ldots$", is greater or equal to i. Hence,

$$\begin{aligned}
\Sigma &\geq 2(y_1 + \ldots + y_{l+k}) + \ldots + 2(y_{i-1} + \ldots + y_{l+k}) + 3(y_i + \ldots + y_{l+k}) \\
&\geq 2(3^{l+k} + \ldots + 3^{l+k-i+1}) + 3^{l+k-i+1} \\
&> 2(3^{l+k} + \ldots + 1), \text{ since } 3^{l+k-i+1} - 1 = 2(3^{k+l-i} + \ldots + 1) \\
&> \kappa(k,l) + \kappa(m,p).
\end{aligned}$$

□

The following lemma is the basis of the inductive proof of [5, Lemma 9.1].

Lemma 5.4. *Let Y and Z be two incomparable subsets of $\{0, \ldots, n\}$. If for every $i \in Y \cup Z$ we have that $x_i \geq 0$, and $\sum_{i \in Y} x_i \leq 3^{|Y|}$, $\sum_{i \in Z} x_i \leq 3^{|Z|}$, then*

$$\sum_{i \in Y \cup Z} x_i < 3^{|Y \cup Z|}.$$

In the sequel, we rely on the following affine transformation that normalises the equations of hyperplanes localised at $\mathcal{B}_1^M$ for

$$M = \big\{\{n, \ldots, 1\}, \ldots, \{n, n-1\}, \{n\}\big\}.$$

Let p be the orthogonal projection from the hyperplane π to the hyperplane $\pi^0 : x_0 = 0$, and let $\mathbf{PA}_n^0$ be the p-image of $\mathbf{PA}_n$. Since p is an affine bijection, the polytopes $\mathbf{PA}_n$ and $\mathbf{PA}_n^0$ are combinatorially equivalent and p maps a face of $\mathbf{PA}_n$ to the corresponding face of $\mathbf{PA}_n^0$.

Consider the $n \times n$ matrices L and L^{-1}

$$\frac{1}{3^n - n - 1} \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \quad (3^n - n - 1) \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

and the vector

$$\mathbf{w} = \begin{pmatrix} 3^n - 3^{n-1} \\ 3^{n-1} - 3^{n-2} \\ \vdots \\ 6 \\ 3 \end{pmatrix}$$

Let $h:\mathbf{R}^n\rightarrow\pi^0$ be the affine bijection mapping $(x'_1,\ldots,x'_n)\in\mathbf{R}^n$ into the point $(x_0,\ldots,x_n)\in\pi^0$, such that $x_0=0$ and

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{w} + L \begin{pmatrix} x'_1-3 \\ \vdots \\ x'_n-3 \end{pmatrix}.$$

Let $u=h^{-1}\circ p$ and let $\mathbf{PA}'_n$ be the polytope obtained as the u-image of $\mathbf{PA}_n$. The polytopes $\mathbf{PA}_n$ and $\mathbf{PA}'_n$ are again combinatorially equivalent.

What are the hyperplanes of $\mathbf{R}^n$ that correspond by u to $\beta^=\cap\pi$, where $\beta\in\mathcal{B}_1^M$? Every such β is of the form

$$\big\{\{n,n-1,\ldots,n-l,\ldots,n-l-k+1\},\ldots,\{n,n-1,\ldots,n-l\}\big\},$$

and let us define the *u-correspondent* of β to be the set $Y=\{n-l-k+1,\ldots,n-l\}$.

The hyperplane $\beta^=$ in $\mathbf{R}^{n+1}$ is of the form

$$x_{n-l-k+1}+2x_{n-l-k+2}+\ldots+(k-1)x_{n-l-1}+k(x_{n-l}+\ldots+x_n)=\kappa(k,l),$$

and let us denote the left-hand side of this equation by Ξ. For Y being the u-correspondent of β, we define the hyperplane $Y^=$ in $\mathbf{R}^n$ by the equation

$$\sum_{i\in Y}x'_i=3^{|Y|},\quad \text{i.e.}\quad x'_{n-l-k+1}+\ldots+x'_{n-l}=3^k.$$

We show that $\beta^=\cap\pi^0$ is the h-image of $Y^=$. By the definition of h, we have that

$$\begin{aligned}
x_{n-l-k+1} &= 2\cdot 3^{l+k-1}+\tfrac{1}{3^n-n-1}(x'_{n-l-k+1}-x'_{n-l-k+2})\\
\vdots\quad & \quad\vdots\\
x_{n-l} &= 2\cdot 3^{l}+\tfrac{1}{3^n-n-1}(x'_{n-l}-x'_{n-l+1})\\
\vdots\quad & \quad\vdots\\
x_n &= 3+\tfrac{1}{3^n-n-1}(x'_n-3),\ \text{and}
\end{aligned}$$

$$\begin{aligned}
\Xi &= 2\cdot 3^{l+k-1}+2\cdot 2\cdot 3^{l+k-2}+\ldots+(k-1)2\cdot 3^{l+1}+k(2\cdot 3^l+\ldots+2\cdot 3+3)\\
&\quad+\tfrac{1}{3^n-n-1}(x'_{n-l-k+1}+\ldots+x'_{n-l}-3k)\\
&= 3^{l+k}+3^{l+k-1}+\ldots+3^{l+1}+\tfrac{1}{3^n-n-1}(x'_{n-l-k+1}+\ldots+x'_{n-l}-3k).
\end{aligned}$$

Hence,

$$\begin{aligned}
\Xi=\kappa(k,l) \quad &\Leftrightarrow\quad \tfrac{1}{3^n-n-1}(x'_{n-l-k+1}+\ldots+x'_{n-l}-3k)=\tfrac{3^k-3k}{3^n-n-1}\\
&\Leftrightarrow\quad x'_{n-l-k+1}+\ldots+x'_{n-l}=3^k,
\end{aligned}$$

which means that $\beta^=\cap\pi^0$ is the h-image of $Y^=$, i.e. the hyperplane $Y^=$ of $\mathbf{R}^n$ corresponds by u to $\beta^=\cap\pi$. The same holds when the superscript "=" is replaced by "$\geq$" in which case the equation $Y^=$ is replaced by the appropriate inequality.

Hence the u-correspondence between the members of $\mathcal{B}_1^M$ and the members of the set

$$\mathcal{B}=\big\{\{n-l,\ldots,n-l-k+1\}\mid 1\leq k\leq k+l\leq n\big\},$$

which is a bijection, is lifted to a bijection between a set of hyperplanes (halfspaces) tied to $\mathbf{PA}_n$ and a set of hyperplanes (halfspaces) tied to $\mathbf{PA}'_n$.

Note that $\mathcal{B}$ is the set of all non-empty connected sets of vertices of the path graph $1-2-\ldots-n$ and it is a building set of $\mathcal{P}(\{1,\ldots,n\})$. It is easy to check that the following implications and its converses hold. If two members of $\mathcal{B}_1^M$ are incomparable, then the u-correspondents are incomparable. If the union of two members of $\mathcal{B}_1^M$ is in $\mathcal{B}_1^M$, then the union of the u-correspondents is in $\mathcal{B}$. If

$N \subseteq \mathcal{B}_1^M$ is 1-nested, then the set of u-correspondents of members of N is a nested set with respect to $\mathcal{B}$. If $N \subseteq \mathcal{B}_1^M$ is a construction of the path graph

$$\{n, \ldots, 1\} - \ldots - \{n, n-1\} - \{n\},$$

then the set of u-correspondents of members of N is a construction of the path graph $1 - 2 - \ldots - n$.

Lemma 5.5. *If $\beta, \gamma \in \mathcal{B}_1$ are such that $\beta^= \cap \gamma^= \cap \mathbf{PA}_n \neq \emptyset$, and β, γ are incomparable, then $\beta \cup \gamma \in C_1 - \mathcal{B}_1$.*

Proof. From Lemma 5.3, we conclude that there is a maximal 0-nested set M such that $\beta, \gamma \in \mathcal{B}_1^M$. By symmetry, we may assume that

$$M = \big\{\{n, \ldots, 1\}, \ldots, \{n, n-1\}, \{n\}\big\}.$$

(In that case x_0 occurs neither in $\beta^=$ nor in $\gamma^=$, since $\beta, \gamma \in \mathcal{B}_1^M$.)

Let $Y, Z \in \mathcal{B}$ be the u-correspondents of β and γ, respectively. Since β and γ are incomparable, Y and Z are incomparable too. Let $a \in \beta^= \cap \gamma^= \cap \mathbf{PA}_n$ and let $a' = u(a)$. Then the coordinates $(x'_1, \ldots, x'_n)$ of a' satisfy the equations $Y^=$ and $Z^=$, and hence, the inequalities $Y^\leq$ and $Z^\leq$.

This means that Y, Z and the coordinates of a', satisfy the conditions of Lemma 5.4 (the coordinates of a' are positive since $x'_i \geq 3$ holds for every $1 \leq i \leq n$). Hence, we have that $Y \cup Z \notin \mathcal{B}$, otherwise, the equation $(Y \cup Z)^\geq$, i.e. $\sum_{i \in Y \cup Z} x'_i \geq 3^{|Y \cup Z|}$ would hold, which contradicts Lemma 5.4. From this we conclude that $\beta \cup \gamma \notin \mathcal{B}_1$, and since both β and γ are in $\mathcal{B}_1^M$, we have that $\beta \cup \gamma \in C_1 - \mathcal{B}_1$. □

As a consequence of Lemma 5.5 and Proposition 3.3 we have the following.

Corollary 5.6. *If $\beta_1, \ldots, \beta_k \in \mathcal{B}_1$ are such that $\beta_1^= \cap \ldots \cap \beta_k^= \cap \mathbf{PA}_n \neq \emptyset$, then $\{\beta_1, \ldots, \beta_k\}$ is 1-nested.*

Proposition 5.7. *For every vertex v of $\mathbf{PA}_n$, there is a 0-face V of C such that*

$$\{v\} = (\bigcap\{\beta^= \mid \beta \in V\}) \cap \pi,$$

and for every $\gamma \in \mathcal{B}_1 - V$, $v \notin \gamma^=$.

Proof. Let v be a vertex of $\mathbf{PA}_n \subseteq \mathbf{R}^{n+1}$. Since the dimension of this space is $n+1$, one concludes that there are $\beta_1, \ldots, \beta_n \in \mathcal{B}_1$ such that

$$\{v\} = \beta_1^= \cap \ldots \cap \beta_n^= \cap \pi.$$

From Corollary 5.6, it follows that $V = \{\beta_1, \ldots, \beta_n\}$ is a 1-nested set, and since it has n elements, by Remark 3.6, it is a 0-face of C. It remains to prove that for every $\gamma \in \mathcal{B}_1 - V$, we have that $v \notin \gamma^=$. This is straightforward, since by maximality of V, we have that $V \cup \{\gamma\}$ is not 1-nested, and by Corollary 5.6, the vertex v cannot be in $\gamma^=$. □

Corollary 5.8. $\mathbf{PA}_n$ *is a simple n-dimensional polytope.*

Proof. This is obvious after Proposition 5.7, or we may apply [5, Proposition 9.7], since every 0-face has exactly n elements. □

Proposition 5.9. *For every 0-face V of C, there is a vertex v of $\mathbf{PA}_n$ such that*

$$\{v\} = (\bigcap\{\beta^= \mid \beta \in V\}) \cap \pi.$$

Proof. We show that for every 0-face V of C, $(\bigcap\{\beta^= \mid \beta \in V\}) \cap \pi$ is a singleton $\{v\}$ such that $v \in \gamma^>$, for every $\gamma \in \mathcal{B}_1 - V$, from which the proposition follows. Again, by symmetry, we may assume that $V = \{\beta_1, \ldots, \beta_n\} \subseteq \mathcal{B}_1^M$, for

$$M = \big\{\{n, \ldots, 1\}, \ldots, \{n, n-1\}, \{n\}\big\}.$$

Let X_i, for every $i \in \{1, \ldots, n\}$, be the u-correspondent of $\beta_i \in V$, for u being the affine transformation introduced after Lemma 5.4. Since V is a construction of

$$\{n, \ldots, 1\} - \ldots - \{n, n-1\} - \{n\},$$

we have that $N = \{X_1, \ldots, X_n\}$ is a construction of the path graph $1 - 2 - \ldots - n$.

By induction on $n \geq 1$, we show that $X_1^= \cap \ldots \cap X_n^=$ is a singleton. If $n = 1$, then $X_1^=$ is $x_1' = 3$, and we are done. If $n > 1$, let L and R be the constructions of the path graphs

$$1 - \ldots - (j-1) \quad \text{and} \quad (j+1) - \ldots - n,$$

such that $N = \{\{1, \ldots, n\}\} \cup L \cup R$. By the induction hypothesis, we have that $\bigcap\{\beta_i^= \mid \beta_i \in L\}$ and $\bigcap\{\beta_i^= \mid \beta_i \in R\}$ are singletons in the corresponding $(j-1)$-dimensional and $(n-j)$-dimensional spaces. Hence, the coordinates $x_1', \ldots, x_{j-1}', x_{j+1}', \ldots, x_n'$ of a point belonging to the above intersection are determined, and it remains to determine the coordinate x_j'. It has a unique value evaluated from the equation $x_1' + \ldots + x_{j-1}' + x_j' + x_{j+1}' + \ldots + x_n' = 3^n$ and the values of $x_1', \ldots, x_{j-1}', x_{j+1}', \ldots, x_n'$.

From this we conclude that

$$(\bigcap\{\beta^= \mid \beta \in V\}) \cap \pi$$

is a singleton $\{v\}$. It remains to show that $v \in \gamma^>$, for every $\gamma \in \mathcal{B}_1 - V$.

Note that in the above calculation of coordinates of $u(v)$, for the jth coordinate we have

$$x_j' = 3^n - (3^{j-1} + 3^{n-j}) > 3^{n-1},$$

since $x_1' + \ldots + x_{j-1}' = 3^{j-1}$ and $x_{j+1}' + \ldots + x_n' = 3^{n-j}$. Note also that $j \in \{1, \ldots, n\}$ was the $\{1, \ldots, n\}$-superficial element with respect to N. In the same way, we may conclude that if m is the X_i-superficial with respect to N, then the mth coordinate of $u(v)$ satisfies $x_m' > 3^{|X_i|-1}$.

For $\gamma \in \mathcal{B}_1^M - V$, let $Y = \{n-l, \ldots, n-l-k+1\}$ be its u-correspondent in $\mathcal{B}$. Since $Y \in \mathcal{B} - N$, it is easy to prove, by induction on n, that there exists $X_i \in N$ such that $Y \subset X_i$ and Y contains the X_i-superficial element with respect to N (cf. [5, Lemma 6.14]). Hence, there is $m \in \{n-l, \ldots, n-l-k+1\}$, which is the X_i-superficial, and $k < |X_i|$. From the preceding paragraph, we conclude that the mth coordinate of $u(v)$ satisfies $x_m' > 3^{|X_i|-1}$, and since all the other coordinates are positive, we have that $u(v) \in Y^>$, i.e.

$$x_{n-l-k+1}' + \ldots + x_{n-l}' > 3^{|X_i|-1} \geq 3^k,$$

hence, $v \in \gamma^>$.

For every other $\gamma \in \mathcal{B}_1 - V$, i.e. when $\gamma \in \mathcal{B}_1 - \mathcal{B}_1^M$, consider the permutohedron P_n obtained as the intersection of the hyperplane π and the halfspaces of the form

$$x_{i_1} + \ldots + x_{i_k} \geq 3^k,$$

where $i_1, \ldots, i_k$ are mutually distinct elements of $\{0, \ldots, n\}$. This permutohedron realises the simplicial complex C_1. The intersection of P_n and the hyperplane

$$x_1 + 2x_2 + \ldots + nx_n = \kappa(n, 0)$$

is an $(n-1)$-simplex. It is the convex hull of the set of points $\{a_1,\ldots,a_n\}\subseteq \mathbf{R}^{n+1}$ (note that $\varepsilon(n)=\frac{3^n-3n}{3^n-n-1}$):

$$\begin{aligned}
&a_1(3^{n+1}-3^n-\varepsilon(n),3^n-3^{n-1}+\varepsilon(n),3^{n-1}-3^{n-2},\ldots,3),\\
&a_2(3^{n+1}-3^n,3^n-3^{n-1}-\varepsilon(n),3^{n-1}-3^{n-2}+\varepsilon(n),\ldots,3),\\
&\quad\vdots\\
&a_n(3^{n+1}-3^n,\ldots,3^3-3^2,3^2-3-\varepsilon(n),3+\varepsilon(n)).
\end{aligned}$$

(The coordinates of a_i are obtained as the solution of the system

$$\begin{array}{lrl}
 & x_1+2x_2+\ldots+nx_n & =\kappa(n,0)\\
 & x_0+x_1+\ \ x_2+\ldots+\ \ x_n & =3^{n+1}\\
(1) & x_1+\ \ x_2+\ldots+\ \ x_n & =3^n\\
(2) & x_2+\ldots+\ \ x_n & =3^{n-1}\\
\vdots & & \\
(n) & x_n & =3,
\end{array}$$

with the equation (i) omitted.)

Since $\gamma\in\mathcal{B}_1-\mathcal{B}_1^M$, the hyperplane $\gamma^=$ is of the form

$$x_{i_1}+2x_{i_2}+\ldots+k(x_{i_k}+\ldots+x_{i_{k+l}})=\kappa(k,l),$$

where $i_k<i_{k+1}<\ldots<i_{k+l}$ and

$(**)$ $i_1,i_2,\ldots,i_{k+l}$ are not consecutive members of the sequence $1,2,\ldots,n$.

Denote the left-hand side of the above equation by Ξ.

We show that for every $i\in\{1,\ldots,n\}$, the coordinates $(x_0,\ldots,x_n)$ of a_i satisfy $\gamma^>$. From this, since v is in the convex hull of $\{a_1,\ldots,a_n\}$, one obtains that $v\in\gamma^>$.

(1) If for some $0\leq m\leq n-l-k$, x_m occurs in Ξ, then

$$\begin{aligned}
\Xi &> x_m > 3^{n+1-m}-3^{n-m}-1=2\cdot 3^{n-m}-1=2(3^{n-m}-1)+1\\
&> \tfrac{3}{2}(3^{n-m}-1)+1=\tfrac{3^{n-m+1}-3}{2}+1\geq \tfrac{3^{l+k+1}-3}{2}+1\\
&> \kappa(k,l).
\end{aligned}$$

(2) If $i_1,\ldots,i_{l+k}\in\{n-l-k+1,\ldots,n\}$, then Ξ is of the form

$$a_1x_{n-l-k+1}+a_2x_{n-l-k+2}+\ldots+a_{l+k}x_n,$$

where $a_m\geq 0$, for every $0\leq m\leq l+k$. Since $(**)$ holds, there is $j\in\{1,\ldots,k-1\}$ such that $a_j>j$ and for every $m\in\{1,\ldots,j-1\}$, we have that $a_m=m$. Therefore,

$$\begin{aligned}
\Xi &\geq x_{n-l-k+1}+\ldots+(j-1)x_{n-l-k+j-1}+j(x_{n-l-k+j}+\ldots+x_n)+x_{n-l-k+j}\\
&=(x_{n-l-k+1}+\ldots+x_n)+\ldots+(x_{n-l-k+j}+\ldots+x_n)+x_{n-l-k+j}.
\end{aligned}$$

For i such that $n-l-k+1\leq i\leq n-l-k+j$, the coordinates $(x_0,\ldots,x_n)$ of the point a_i satisfy the inequality

$$(x_{n-l-k+1}+\ldots+x_n)+\ldots+(x_{n-l-k+j}+\ldots+x_n)>3^{l+k}+\ldots+3^{l+k-j+1}.$$

For the other values of i, the above inequality converts into equality. Also the $(n-l-k+j)$th coordinate $x_{n-l-k+j}$ of a_i is greater than $3^{l+k-j+1}-3^{l+k-j}-1$.

Hence,

$$\begin{aligned}
\Xi &> 3^{l+k-j+1}\frac{3^j-1}{2} + 3^{l+k-j+1} - 3^{l+k-j} - 1 \\
&= \frac{3^{l+k+1} - 3^{l+k-j+1} + 2\cdot 3^{l+k-j+1} - 2\cdot 3^{l+k-j}}{2} - 1 \\
&= \frac{3^{l+k+1} + 3^{l+k-j}}{2} - 1 \\
&\geq \frac{3^{l+k+1} + 3^{l+1}}{2} - 1, \quad \text{since } k \geq j+1 \\
&> \frac{3^{l+k+1} - 3^{l+1}}{2} + 1 > \kappa(k,l).
\end{aligned}$$

□

Proof of Theorem 5.2. From Propositions 5.7, 5.9 and Corollary 5.8, we conclude that the face lattice of $\mathbf{PA}_n$, with bottom removed, is given by

$$\Big(\bigcup\big\{\mathcal{P}(\{\beta^{=} \mid \beta \in V\}) \mid V \text{ is a 0-face of } C\big\}, \supseteq\Big),$$

from which the theorem follows. □

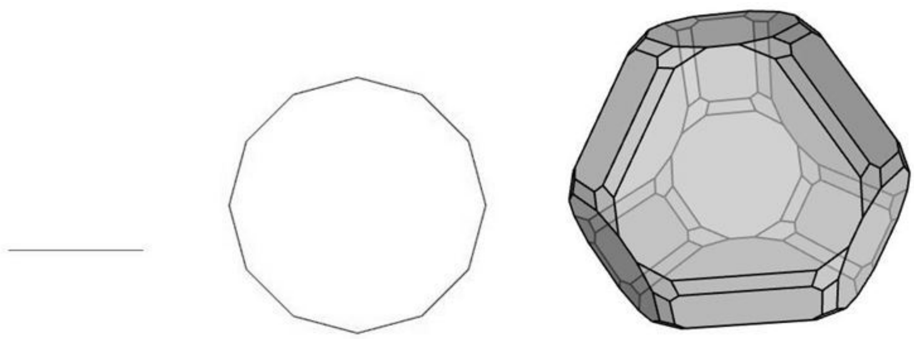
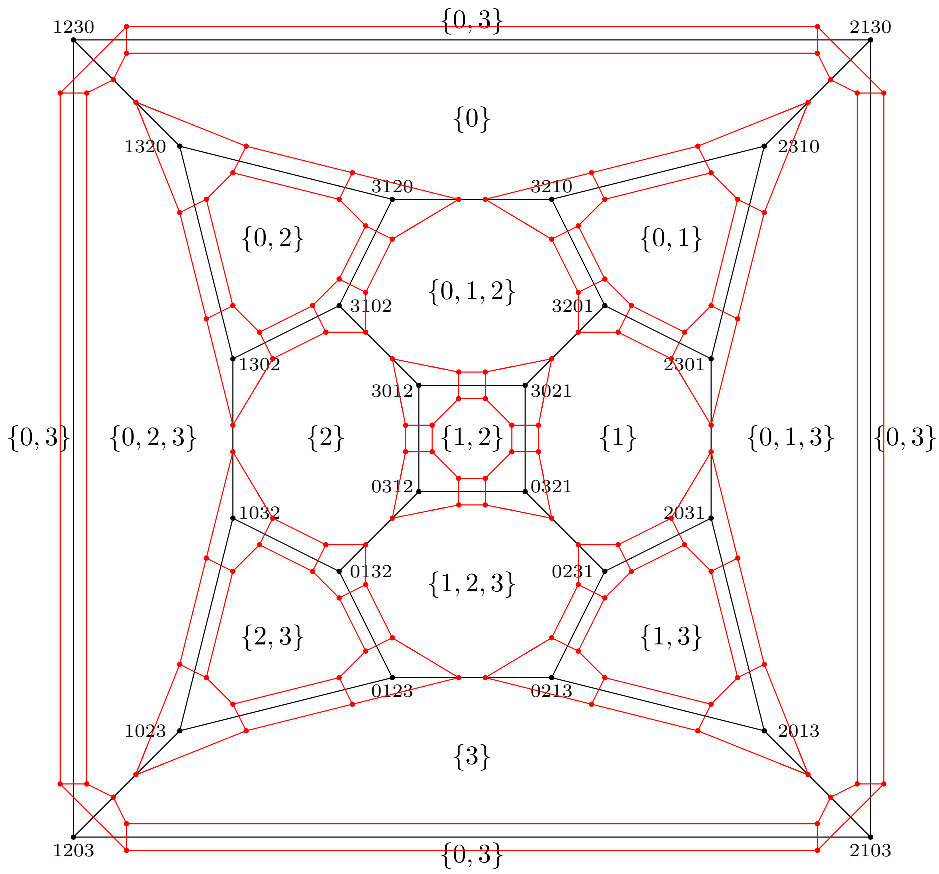


Figure 2. $\mathbf{PA}_1$, $\mathbf{PA}_2$ and $\mathbf{PA}_3$

Figure 2 illustrates the n-dimensional simple permutoassociahedra for $n = 1, 2$ and 3. We conclude the paper with a picture that indicates the connection between the 3-dimensional simple permutoassociahedron and the 3-dimensional permutohedron. The facets of the permutohedron are labelled by the elements of

$$\mathcal{B}_0 = \mathcal{P}(\{0,1,2,3\}) - \{\{0,1,2,3\}, \emptyset\}.$$

As it has been already noted, it is more intuitive to transform every such $A \in \mathcal{B}_0$ into the ordered partition of $\{0,1,2,3\}$, with $\{0,1,2,3\} - A$ as the first block and A as the second block.



A vertex of the permutohedron is labelled by the permutation corresponding to the maximal 0-nested set, i.e. to the set of all the facets incident to this vertex. The facets of the permutoassociahedron correspond to the elements of $\mathcal{B}_1$. For example, the pentagon that covers the vertex labelled by 2301 corresponds to the set $\{\{0, 1, 3\}, \{0, 1\}, \{1\}\}$.

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