FORM-FINDING OF SHELL STRUCTURES BASED ON ISOGEOMETRIC ANALYSIS

TRAŽENJE FORME LJUSKI ZASNOVANO NA IZOGEOMETRIJSKOJ ANALIZI

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SUMMARY

The paper reviews potentials of the application of isogeometric analysis (IGA) in the processes of form-finding of shell structures. Shaping of those specific spatial structural typologies is determined by consideration of the form-structure interrelation, therefore the sustainable approach to design implies application of form-finding techniques. In this paper, we applied the frequently used form-finding principle a inverted model strategy, more precisely concept of inverted displacements. Contrary to the similar studies which apply standard finite element (FE) discretization, we proposed an application of isogeometric modeling of the structure. Isogeometry implies analysis based on the exact definition of the initial geometry, facilitating design and analysis by application on the single data type – model. Advantage of the conservation of geometric representation is especially evident in the case of shell structure designs, bearing in mind their continuously curved geometry. We applied NURBS based isogeometric analysis in which geometry and displacement field are described by NURBS surfaces. Modeling of the structural performances was conducted by isogeometric shell elements. Presented numerical experiment has a function to confirm the effectiveness of the proposed approach.

Key words: *Shell structures, Shell design, Form-finding, Isogeometric analysis, NURBS surface.*

REZIME

Radom se razmatraju potencijali primene izogeometrijske analize u procesima traženja formi ljuski. Oblikovanje ovih specifičnih prostornih strukturalnih tipologija određeno je razmatranjem interrelacije forma-struktura, zbog čega održivi pristup njihovom projektovanju podrazumeva primenu tehnika traženja forme. U okviru ovog rada primenjena je strategija obrnutog modela, često korišćen princip traženja forme, tačnije koncept obrnutih pomeranja. Za razliku od sličnih istraživanja u kojima se koristi standardna diskretizacija konačnim elementima (FE), ovim radom se predlaže izogeometrijsko modelovanje strukture. Izogeometrija podrazumeva analizu zasnovanu na egzaktnoj definiciji početne geometrije, čime je omogućeno projektovanje i analiza upotrebom jedinstvenog tipa podataka – modela. Prednost konzervacije geometrijske reprezentacije posebno dolazi do izražaja u slučaju projektovanja ljuski imajući u vidu njihovu kontinualno zakrivljenu geometriju. Primenjena je NURBS izogeometrijska analiza, u kojoj se geometrija i polje pomeranja opisuju korišćenjem NURBS površi. Modelovanje strukturalnih performansi izvršeno je korišćenjem izogeometrijskog elementa ljuske. Prikazan numerički eksperiment u funkciji je potvrđivanja efikasnosti predloženog pristupa.

Ključne reči: *ljuske, projektovanje ljuski, traženje forme, izogeometrijska analiza, NURBS površ.*

1. INTRODUCTION
1. UVOD

In the design of shell structure configurations, the external loads, internal forces and displacements of the structure must be considered simultaneously in three dimensions. The sustainable approach for the shaping of those continuous surface systems is by the application of form-finding techniques.

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For a long time, the form-finding was exclusively done through construction of the precise scale physical models. Those physical experiments remain important and useful medium of the design exploration in the early stage of the design process and education, facilitating architects to be directly included in the design process of surface structures. The application of physical models was suppressed by the development of the computational analyses, especially using finite element analysis (FEA). Computational form-finding tools usually simulate behavior of physical models, striving to overcome restrictions of manual experiments by the application of more efficient, reliable and precise computational simulations.

Independently from the applied medium, the form-finding can be used for conceiving and testing of design solution, comprehension of load transfer, estimation of the stresses and deformations. The form-finding is an iterative process in which structurally rational forms are obtained through incremental adjustments of form, forces, supports, material, thickness, etc.

In this paper, we introduced the form-finding of architectural shell structures based on the isogeometric analysis (IGA), focusing on the advantages that are related to the use of this relatively recent numerical approach. The isogeometric concept proposed by Hughes *et al* (2005) is toward integration of CAD and CAE models. Integration is realized by conservation of geometric representation, enabling design and analysis of models by using single data type (exactly defined geometric representation), maintaining continuity of design-analysis cycle, facilitating and accelerating the process of repeated analysis. Advantage of the isogeometry is especially evident in the case of shell structures, because of their continuously curved geometry.

In this study, we applied the inverted displacement strategy. This is the one of the frequently used form-finding strategies that applies inverted model principle. The simulation was conducted by NURBS based isogeometric analysis. In this case both geometry and displacements were represented by NURBS surfaces. Modeling of the structural performances was conducted using the isogeometric Kirchhoff shell elements formulated by Radenković (2014).

The proposed approach was tested by a numerical experiment. The results confirm effectiveness of the proposed approach and potentials of its application in the design processes.

2. INVERTED MODELS
2. OBRNUTI MODELI

Form-finding processes are commonly based on:
– principle of the inverted hanging model and
– concept of stress control.

The both principles could be realized by physical and computational tools.

The forms of the first thin shells built in the beginning of the XX century are realized mainly by the application of the simple geometric shapes (spheres, cylinders, parabolic shapes, hyperbolic paraboloids, rotational hyperboloids, etc.) and known relations between span and curvatures. Although development of lightweight structures in certain way stimulated engineers to investigate, the application of the simple forms was partly due to the fact that the calculations could be only conducted for the analytically defined geometries. However, those geometries cannot provide exclusively membrane state of stress, causing the need to introduce additional elements such as edge beams, diverse kind of stiffeners, prestressing, etc.

The hanging models are based on the specific behavior of catenary, which were applied for the design of the concrete shells from the middle of the XX century. They are used in the simulation of hanging structures, which shape emerges as a response to the magnitude and the position of the forces acting on them. Contrary to the common procedure in which a structure is determined for the previously defined shape the inverted hanging model procedure implies that: *assigning several geometric parameters such as span, height, load and desirable stress and displacement constraints, initiates finding of the natural equilibrium shape*. Shells achieve the stiffness only if their form provides membrane state of stress that is in an equilibrium with the external forces.

For example, the chain hanged between two points loaded by the concentrated force in the middle position will take "V" shape, while under the equally distributed load it will take parabolic shape. While the hanged chain represents tensioned element, an arch obtained by the inversion represents a compression structure. Placing chains in two directions results in a cable net that approximates surface, and its inversion gives vaults and shells. In this systems the impact of the self-weight does not cause banding, influencing efficient use of the material and facilitating application of thin structural elements, frequently without the need for presterssing (Figure 1).

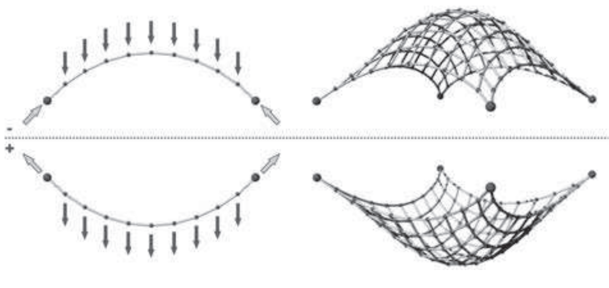

Figure 1. Illustration of the principle of the inverted hanging model

Slika 1. Ilustracija principa obrnutog visećeg modela

Numerous examples (Ramm, 1991), (Zalewski and Allen, 1998), (Ramm, 2004), (Bechthold, 2008), (Basso, 2011), (Bletzinger, 2011) have demonstrated that the resulting shapes are natural, elegant, and favorable stress state eliminates the necessity for additional stiffeners.

The application of the inverse hanging chain in the process of definition of the ideally stressed forms of arches and vaults is known since Hooke anagram *riddle of arch* from the XVII century (Truesdell, 1960). In that time structures designed by this principle were built by available materials. Several prominent constructors applied this principle for realization of their ideas: Wren for the St. Paul's Cathedral in London, Poleni for model of rehabilitation of the St. Pethers Cathedral in Rome, Rondelet for Dome of French Pantheon in Paris, Hübsch in his works (Tomlow *et al.* 1989), etc. The famous examples are chain models that Gaudi used for the construction of his daring architectural creations. By this technique he built a three-dimensional hanging model for the crypt of *Colònia Güell* in Barcelona (Tomlow *et al.* 1989).

The contribution of Hains Isler in research of form-finding techniques includes their application in the design process of his constructions, proposition of the innovative approaches and improvement of various techniques (Isler, 1959), (Isler, 1980), (Billington, 1980), (Chilton, 2000), (Ramm, 2011), (Billington, 2011).

Frei Otto gave outstanding contribution to the evolution of the form-finding by his experimental works which a conducted with the team of researchers at *Institut für Leichte Flächentragwerke* – IL, at Stuttgart University. In his researches he experimented with diverse form-finding techniques including hanging models (Otto and Rasch, 1996), and realization of several buildings confirms the effectiveness of this design approach. The example is the grid shell *Multihalle* in Mainheim (IL, 1978), which form was obtained by construction of hanging model.

The development of computational technologies suppressed the application of physical models in design of shell structures. Computational models usually simulate behaviors of physical models. They overcome restrictions of manual experiments by the application of more efficient, reliable and precise simulations. Since 1960s numerous computational methods were developed (Ramaswamy, 1991), (Adriaenssens *et al*, 2014), (Veenendal and Block, 2012), (Milošević, 2015) including methods that simulate inverted hanging models. The application of FEA in form-finding has been presented in (De Veubeke, 1965), (De Veubeke, 1968). Also, numerous tools for physics based modeling were developed in the field of computer graphics from the end of 1990s including tools which enable free form modeling (Celniker and Gossard, 1991), (Terzopoulos and Qin, 1994). The advantages of the computational tools, constant advancements and their transformation from the specialized tools to some more accessible resources have attracted a significant interest of designers.

The production of the free-form architectural designs in the last years re-actualizes the subject of the design potentials of hanging chain principle in the context of structural and formal aspects of the designs obtained through form-finding process as a rational alternative to the un-contextual geometrical sculpturing by the application of digital technologies. Regardless the mediums used for simulations obtained results facilitate an approximation (proportions, space, height, etc.) and evaluation of quality of designs in the aesthetic, spatial and to some extent structural terms.

3. ISOGEOMETRIC ANALYSIS AS A FORM-FINDING TOOL
3. IZOGEOMETRIJSKA ANALIZA KAO ALAT ZA TRAŽENJE FORME

The FEA is usually applied for the stress and strain evaluation from the middle of the XX century (Hughes, 2000). This high reliable numerical method can be also used for form-finding (Bechthold, 2008). The IGA became relatively recent a branch of numerical analyses (Hughes *et al.*, 2005). The isogeometric concept implies the use of the same mathematical description for both geometry (CAD) and analysis (CAE) model of a structure. In this paper, we proposed application of IGA for form-finding of shell structures.

The principal advantage of application of the IGA in form-finding is conservation of geometric representation, implying simplification of the modeling, increasing of accuracy and preservation of the design-analysis cycle.

The standard FEA is based on a mathematical and physical discretization. The physical discretization is a replacement of a real system with a discrete model which is composed by an adequate number of finite elements (FE) of corresponding shape, type and arrangement. In the case of complex architectural forms the generation of the FEA model based on the CAD geometry becomes more difficult task than performing the analysis. IGA simplifies the construction of the analysis model replacing the classical FEA procedure with procedure based on the CAD geometry. Broadly speaking, diverse technologies of geometric representation could be used in IGA. In this paper, we used NURBS technology because of its currently widespread application in the design.

A discrete model does not represent an accurate replica of the CAD geometry, but segmented polynomial approximation. While in the classical FEA the geometry is described by a typical low-order interpolation, IGA allows the use of the original CAD basis. Instead of application of the finite element model that approximates geometry the application of the native geometry provides exact representation. The applied geometry enables the fulfillment of compatibility and stability (convergence) criteria. The most important advantage of the IGA is that in numerical tests it demonstrates to be more accurate in comparison to the classical FEA.

The isogeometric concept ensures integration and continuity of design-analysis cycle. Integration of the CAD and CAE models was an elusive goal due to the discretization. The construction of the analysis model from the design model in the preprocessing phase of the classical FEA produces deviations. Isogeometric approach provides consistency in the modeling process, facilitates repeated analysis in the case of parametric and free-form changes of geometry and optimization.

2.1 IGA shell element
2.1. IGA element ljuske

In NURBS based IGA numerical modeling of the displacement fields is done by NURBS bases.

NURBS is a general tool for parametric geometric modeling that facilitates efficient and precise representation of diverse geometric forms using minimal amount of data, simple form control and intuitive shaping (Piegl and Tiller, 1997), (Farin, 1999), (Rogers, 2001). NURBS control points and associated base functions define geometry (Figure 2).

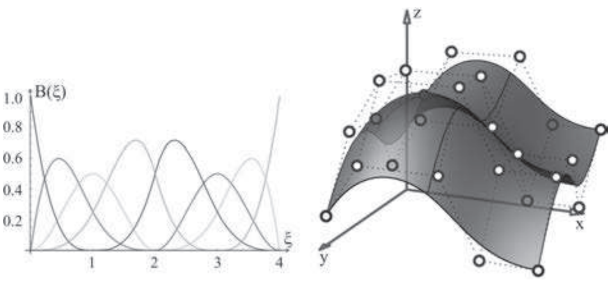

Figure 2. Cubic B-Spline basis function with open knot vector (left), and NURBS surface with its control polygon (right)

Slika 2. Kubna B-Spline bazna funkcija na otvorenom čvornom vektoru (levo), i NURBS površ sa svojim kontrolnim poligonom (desno)

NURBS functions fulfill the required properties for application in the finite element formulations (Cottrel *et al*, 2009), (Hughes and Evans, 2010). In contrast to classically used shape functions, they are generally non-interpolating (Figure 2), and imply two meshes which represent decomposition of the real geometry: (1) control mesh and (2) physical mesh. There are two sorts of elements in the physical mesh: (1) macro element – patch and (2) micro element – knot span. Any macro element can be decomposed into the micro elements.

The non-interpolating characteristic of NURBS becomes problematic when sudden changes of low continuity, like in kinks, are the part of the shape that should be modeled. In the aim of achieving the greater precision, basis functions can be refined by an insertion of control points and elevation of the degree with no influence on the original geometry and its parameterization (Cottrel *et al*, 2007). The techniques of mesh refinement can be used without connection to the CAD data basis, which is not possible in the classical FEA.

Considering that the same functions are applied for geometric representation and analysis this approach is isoparametric. Boundary conditions are applied on control points and control variables. IGA represents generalization of classical FEA, i.e. it is also Galerkin numerical method. NURBS geometry has convex hull property. The system of equation is sparse linear. Abstraction of the system by NURBS structure practically enables presentation that has unlimited possibilities and freedom of geometric presentation (single-patch, multi-patch).

The classical theory of the surface systems is based on the Kirchhoff hypothesis. It represents basis for the derivation of conclusions about kinematics and deformations. In the classical theory degrees of freedom (DOFs) are displacements and rotation. Application of the NURBS basis for the description of the surface system enabled formulation in which only DOFs are displacements.

The isogeometric shell element is formulated starting with the kinematic description of the deformation (Figure 3). In this formulation DOFs are control variables. An advantageous property of the shell element is their load bearing, exclusively via membrane forces. Therefore, only description of the displacements *u* and the resulting strains ε is necessary.

A range of isogeometric shell elements formulations has been developed (Kiendl *et al*, 2010), (Dornisch *et al*, 2013).

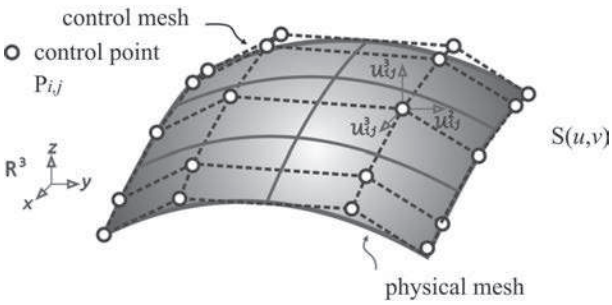

Figure 3. Kirchhoff rotationless shell finite element

Slika 3. Kirhofov konačni element ljuske bez rotacija

Related to the applications in the form-finding of shell structures the self-weight is crucial load case. This load case is usually applied in the processes of form-finding of rigid surface structural systems since it represents a dominant case, while modeling of other load cases is less frequently used for this purposes (Batchlod, 2008). In the case of concrete shells this represents nearly real situation. However, in the case of considerable environmental influences (e. g. snow or wind loads) or in the case when the influence of a the certain load is dominant, to obtain an optimal shape it is necessary to include influences of those loads in form-finding procedure.

2.2. Algorithm of the form-finding procedure based on IGA
2.2. Algoritam procedure traženja forme zasnovane na IGA

The proposed procedure is an iterative form-finding process of shell structures summarized by the flowchart in Figure 4. In the proposed algorithm, the form-finding is realized through the modifications of the initial geometry based on the application of the displacement values from the first analysis step. The process starts by the isogeometric analysis of the original surface and in the second step we read the displacements of the original system and modify its geometry by inverting and scaling up displacement vector.

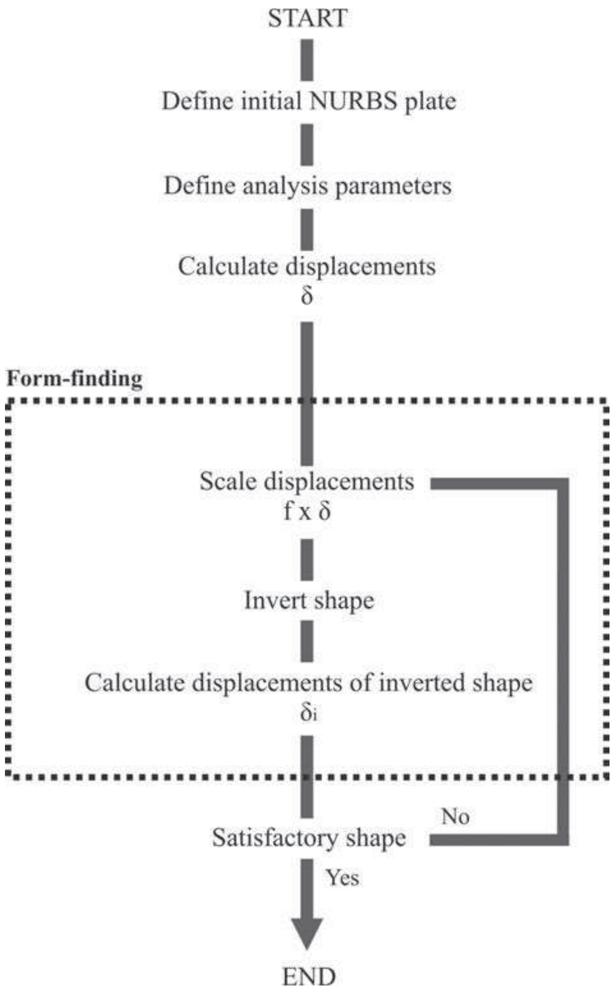

Figure 4. Flowchart of the form-finding procedure based on IGA

Slika 4. Dijagram procedure traženja forme zasnovan na IGA

The form-finding process can be summarized as follows. Starting from the definition of the initial NURBS surface geometry, boundary conditions, and material parameters we perform IGA to calculate displacements of the original geometry δ. In the next step, we modify geometry by scaling displacements of the original geometry $f \cdot \delta$ and invert the modified shape. In order to determine structural behavior of the modified shape, we calculate its displacements δ_i. The process repeats until we obtain the satisfactory shape.

4. NUMERICAL EXPERIMENT
4. NUMERIČKI EKSPERIMENT

The numerical experiment illustrates application of the computational form-finding method based on isogeometric analysis demonstrating potential for its application in architectural design of shell structures. In this context two aspects are of major interests:

– Advantages of the computation including the exact NURBS geometry. Since form-finding of the smoothly curved shell structures should be repeated several times, a fast computation due to a very low number of necessary control points is attractive.

– Advantage that the common model for the design and analysis stays intact during form-finding. This allows for modifications based on the resulting shape in order to iteratively approach a shape that fulfills functional, structural and aesthetical requirements.

4.1. Materials and methods
4.1. Materijali i metode

Based on information of the structural performances the objective of the form-finding process is to find the satisfactory shape through the process of the modification of the initial geometry.

The initial geometry in this experiment is a 20 m square flat plate. The thickness of the plate is 15 cm. The initial geometry is modeled using 3 finite elements with a third-degree function in both directions of the plate. This gives 9 finite elements in total, 16 control points for each finite element and 48 DOFs for each element. Also, it should be pointed out that this example includes only linear static analysis.

In order to calculate displacements of the initial geometry, we define boundary conditions and material parameters. The structure is supported in four corner points and loaded with the self-weight. Parameters of the applied material are: Young's modulus E=48 MPa, Poisson ratio ν=0.2 and density ρ=2500 kg/m^3.

In the next step the form-finding, based on the concept of the inverted displacements, was conducted. By inverting directions of the displacements, we obtain deformed geometry which shape depends on boundary conditions, i.e. instead of the flat surface it has much more stable vault shape. Obtained deformed geometry was adopted as starting point for the next step of the analysis. The modification of geometry (form-finding) was realized by scaling the displacement values of the initial geometry (flat surface). We used following scaling factors: 0.5, 1.0, 1.5, 2.0, 2.5, and 3.0. By scaling displacement values, we increase the stiffness of the structure.

For the computation of displacements, we used isogeometric shell element. All structural computations have been performed using an IGA open-source code written in Wolfram Mathematica® 10 (Wolfram, 2015).

4.2. Results and discussion
4.2. Rezultati i diskusija

In this case the form-finding is process of deformation of the starting flat geometry, representing topology of the surface, which results are doubly curved vaults (Figure 5). The shape of the structure represents function in respect to the boundary conditions and loads. Generally speaking, in form-finding processes manipulation of the shape can be realized by the change in the distribution and magnitude of the loads, position of supports and the

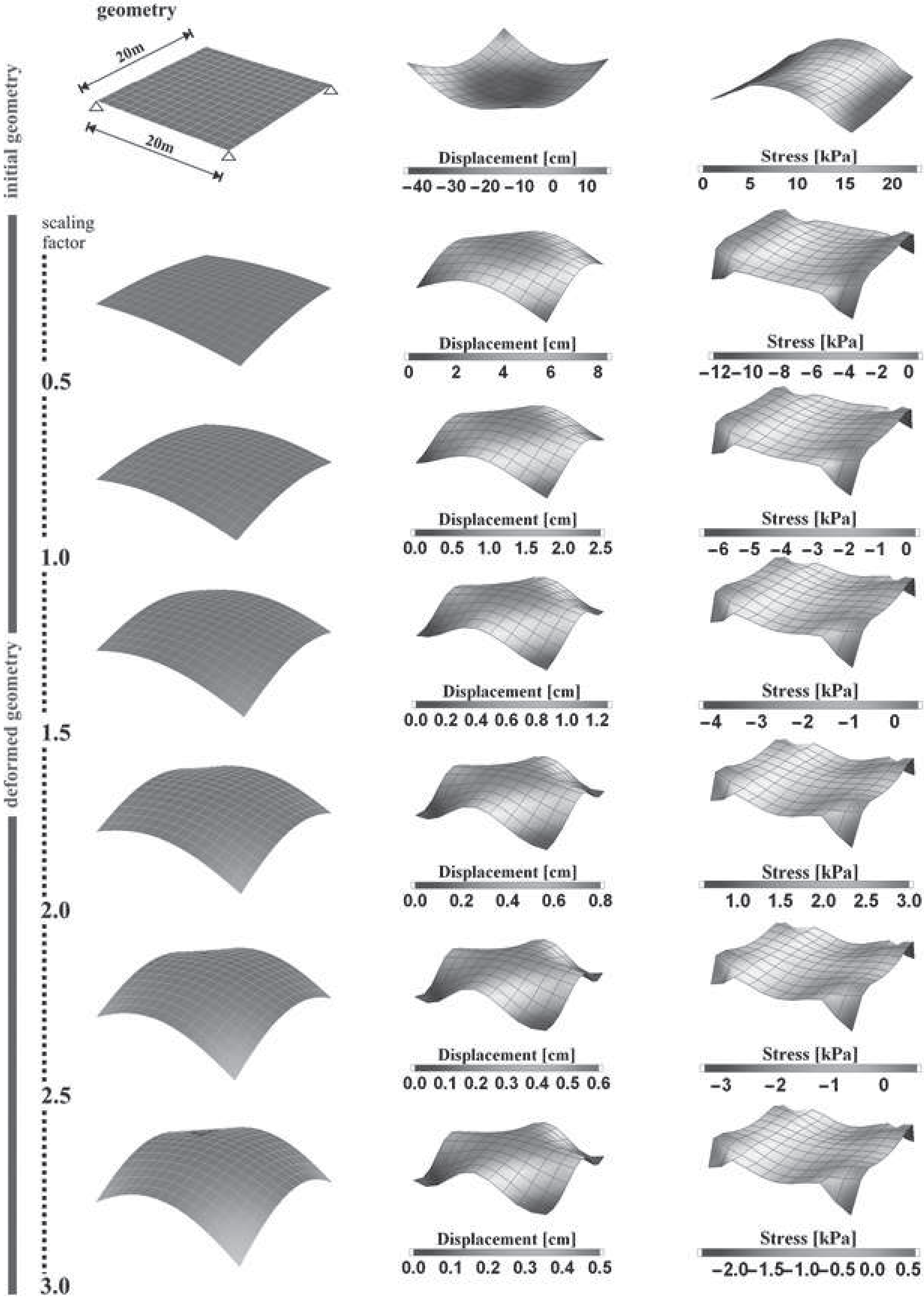

Figure 5. Preview of the variant forms obtained by application of the inverted displacement principle
Slika 5. Prikaz varijantnih formi dobijenih primenom principa obrnutih pomeranja

geometry. In this case, the deformation of the surface is realized by varying scaling factor of displacements. Using this, we manipulate with the curvature of the structure which produces shape that has better structural properties.

The numerical experiment also confirmed advantage of the approach in which shell geometry is modeled by NURBS surface technology. A limited number of control points when using NURBS as basis functions results in the low number of DOFs which implies the faster computation without losing quality in the geometry description. By using NURBS as basis for the geometry description throughout the form-finding the complete and continuous mathematical description of the shell shape is conserved.

In terms of architectural design, a continuous description of the surface allows precise and fast visualization of the form-finding results. This is especially advantageous in design of architectural shell structures which must satisfy functional, technical, aesthetical requirements. Curvature-related properties are often major design goals, since they have an important influence on the structure appearance (visual effect), structural performances and functional requirements. In design, based on the conventional FE discretization, the evaluation of curvature change may be quite challenging, since no direct description of the geometry exists. With the presented isogeometric framework and related continuous surface description, an evaluation of the curvature or any other surface property is straightforward and it is directly connected to the surface description.

The design experiment confirmed advantage of the application of the integration of design and analysis in one single environment enabled by isogeometric concept. This is especially favorable since CAD and CAE use common geometry description enriched by the respective specific data. Abstraction of the system by application of NURBS technology has practically unlimited possibilities and freedom of geometric presentation.

Assigning the physical (material) attributes to the geometric substrate and using the functions for the geometry description directly in the analysis, facilitates application of the single model based on geometric representation through the entire process. Any modification in the geometry is straightforward integrated in further design and analysis steps. The integration of the design and analysis enabled by isogeometric concept is especially advantageous in surface systems design. In the case of design of shell structure this aspect is of special interest since aesthetical and mechanical decisions cannot be separated. Their design usually takes several iterations until technical and aesthetical requirements are finally satisfied. In this case manual update of various models is quite time consuming. Respectively, working in common framework is quite attractive.

5. CONCLUSION
5. ZAKLJUČAK

Computational form-finding experiments represent a useful tool that supports designs in the conceptual phase and the medium of communication between architects and engineers in the collaborative process. Though IGA has far greater possibilities, this study demonstrates that it can be also used as a form-finding tool.

In the design of shell structures, one of the major advantages introduced by application of IGA is possibility to use the same exact geometry representation. Herein presented IGA based form-finding is focused on the studies of the deformed surfaces, and represents a part of the incremental process of improvement of geometry of structure. The design of architectural shells is an iterative procedure, looping through variation of parameters (e. g. different boundary conditions or loads). Even in the case of form-finding, applied in the phase of conceptual design, that does not have to result with the definition of the final geometry, but approximation which will be refined through a detailed structural analysis in the phase of design elaboration, advantages of application same geometric representation is evident.

Another advantage of the IGA modeling is using single environment, contrary to the approach in which geometry was modeled in CAD and analysis conducted in CAE software. Full potential of this method will be realized by implementation in CAD environment, because modeling in the CAD systems might be easier compared to many of the known preprocessors. Parametric CAD modeling environments are especially interesting in this respect, because they could enable a simple modification and adjustment of the surface geometry, changes in the surface thickness, supports, material parameters, during the work process, which considerably facilitates production of variant solutions necessary in the design processes based on the information from the structural analysis.

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