SPATIUM *International Review*
No. 22, July 2010, pp. 67-72

UDC 004.432.41MATHEMATICA:72.012
Preliminary report-Short communications

PARAMETRIC CURVES AND SURFACES: MATHEMATICA DEMONSTRATIONS AS A TOOL IN EXPLORATION OF ARCHITECTURAL FORM

Ljiljana Petruševski[1], University of Belgrade, Faculty of Architecture, Belgrade, Serbia
Milana Dabić, University of Belgrade, Faculty of Architecture, Belgrade, Serbia
Mirjana Devetaković, University of Belgrade, Faculty of Architecture, Belgrade, Serbia

Through their work in the field of generic explorations, authors of this paper have developed a series of demonstration projects, based on the software package MATHEMATICA, related to the parametric representation of curves and surfaces in space. The main purpose of the demonstrations is exploration of generic potential of curves and surfaces in order to apply it to the process of generating architectural form. The paper presents the developed demonstration projects and their application within the generic explorations of architectural form.

Key-words: Mathematica demonstrations, parametric curve and surface, architectural form

INTRODUCTION

Wolfram Demonstration Project has been developed by Wolfram Research software company. The aim of this project is to educate in science, technology, mathematics, art, finance and other realms[2]. With help of Mathematica, one of the most powerful computer program for calculus, wide auditorium got chance to illuminate many scientific concepts. Project is growing daily, because the Mathematica users are allowed to make and upload demonstrations, which could be used with free software – Mathematica Player.

The development of Mathematica has been started by Stephen Wolfram. His idea was to build a computer program for simulating natural phenomena. Unexpectedly simple algorithms gave results, which initiated the exploration of generic forms (Wolfram, 2002). Since the idea of generic concept is in basis of Mahematica, there are many demonstrations which illustrate the most important ones: fractals, L-systems, cellular automata (Bogdanov et al., 2007; Petruševski et al., 2009; Devetaković et al., 2009).

[1] Bul. kralja Aleksandra 73/II, 11 000 Beograd, Serbia
ljpetrusevski@sbb.rs

[2] Wolfram Demonstrations Project

As a support for their own exploration[3] and to help their students within the courses Mathematics in Architecture[4] and Generic Explorations[5], Faculty of Architecture, University of Belgrade, the authors of this paper have developed series of Demonstration projects[6] which graphically represent parametric curves and surfaces. The main purpose of these demonstrations is exploration of generic potential of curves and surfaces, in order to apply it to architectural form exploring and generating process.

PARAMETRIC CURVES AND SURFACES

One way of representing curves and surfaces, in Cartesian coordinate system, is parametric.

A planar curve *L* is a map of interval I in two-dimensional space ($L: I \to R^2$). For each $t \in I$

$$L(t)=(x(t),y(t)), \qquad (1)$$

[3] Web resource: Generic explorations

[4] Web resource: Mathematics in Architecture

[5] Web resource: Generic explorations, elective course

[6] Web resource: Demonstration projects by Milana Dabić

where $x: I \to R$, and $y: I \to R$ are real-valued functions (Shelden, 2002).

If we choose Cartesian coordinate system for two-dimensional space representation, then L(t)=(x,y), represents a set of points with coordinates

$$x=x(t),\ y=y(t),\ t \in I \qquad (2)$$

These equations are called parametric representation of curve *L* (Pottmann et al., 2007).

Example: Ellipse in Figure 1a) is represented with equations:

$$x=5\cos(t),\ y=2\sin(t),\ t\in[0,2\pi] \qquad (3)$$

Ellipse in Figure 1b) is represented with equations:

$$x=2\cos(t),\ y=5\sin(t),\ t\in[0,2\pi] \qquad (4)$$

A space curve *L* is a map of interval I in three-dimensional space $L: I \to R^3$,

$$L(t)=(x(t),y(t),z(t)),\ t \in I \qquad (5)$$

where *x*, *y* and *z* are real-valued functions,

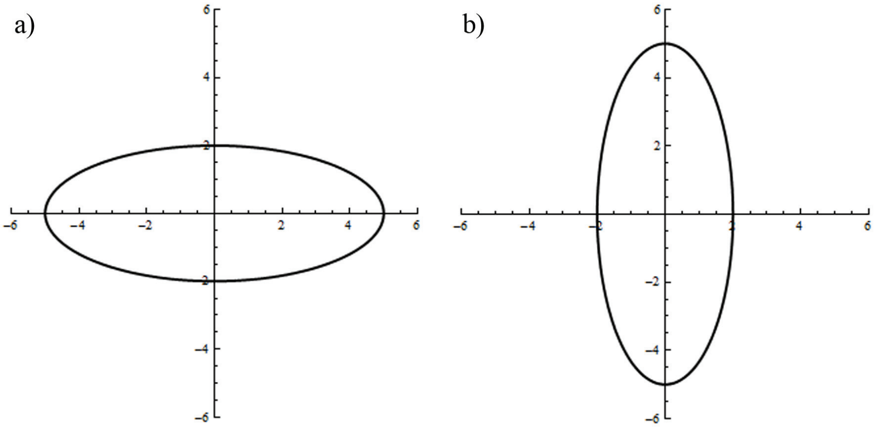

Figure 1 - Parametric representation of ellipse

$x: I \to R$, $y: I \to R$, $z: I \to R$. Points of space curve *L* have coordinates *L(t)=(x,y,z)*, hence equations

x=x(t), y=y(t), z=z(t), $t \in I$ (6)

are parametric equations of space curve *L*. One could notice that parametric equations of planar and space curves are functions of one variable *t*.

A surface *S* is a map of $I \times J$ in three-dimensional space: ($I \times J \to R^3$), where $I \times J$ denotes Cartesian product of intervals *I* and *J*. For each $u \in I$, $v \in J$,

S(u,v)=(x(u,v),y(u,v),z(u,v)) (7)

where *x*, *y* and *z* are real-valued functions of set $I \times J$. Coordinates of surface points *S(u,v)=(x,y,z)*, are functions of two variables, described with equations:

x=x(u,v), y=y(u,v), z=z(u,v), (8) $(u,v) \in I \times J$.

These equations are called parametric representation of surface *S*.

Parametric curves could be discretized by finding the polyline (polygonal curve) that represents curve. For surfaces, it is desirable to cover it with planar panel elements and use polyhedral surfaces (Pottmann et al., 2006). One could also control how precise that representation could be. As higher number of line segments or panels is, the curve or surface is better represented.

In parametric representation of curves and surfaces, it is possible to replace constants with additional parameters. Thus, collection of curves or surfaces, is created, which have different shape or position for various values of these parameters. In Example, by adding parameters *a* and *b* instead of constants, collection of ellipses is created

x=acos(t), y=bsin(t), $t \in [0, 2\pi]$, (9)

which is for parameter values *a=5*, *b=2* shown in Figure 1a), and parameter values *a=2*, *b=5* in Figure 1b).

One of the challenging issues in understanding parametric representation, is seeing parametric object as static set of points, without noticing connection between functional relationship of parameter values and assigned points (Filler, 2007). Solving this problem was one of the motives for authors of this paper to develop parametric curves and surfaces demonstrations.

WOLFRAM DEMONSTRATIONS FOR REPRESENTIG CURVES AND SURFACES

The authors of this paper have developed following projects as a tool for generic explorations: *"Circle, Ellipse, Hyperbola, and Astroid"*, *"Cycloid and Archimedes's Spiral"*, *"Looped Curves"*, *"Four Space Curves"* and *"Parametric Representations of Four Surfaces"*[7]. Paper describes demonstrations: *"Looped Curves"*, *"Four Space Curves"* and *"Parametric Representations of Four Surfaces"*, illustrated respectively in Figures 2, 3 and 4.

In the above mentioned demonstrations several geometric objects are presented. After one of them is chosen, parametric equations are displayed. Also, there are sliders for choosing parameter values in equations. With change of parameter values, graphic preview of object changes simultaneously, so one can notice how particular parameter affects graphic interpretation. This way, geometric object is not seen as a static, but as a dynamic object, which improves understanding of parametric representation.

Figure 2 presents *"Looped Curves"* demonstration with chosen curve

x=asin(k1t)cos(k2t), (10)
y=bsin(k1t)sin(k2t), $t \in [0, 2\pi]$

for parameter values *a=5*, *b=5*, $k_1=3$ and $k_2=5$.

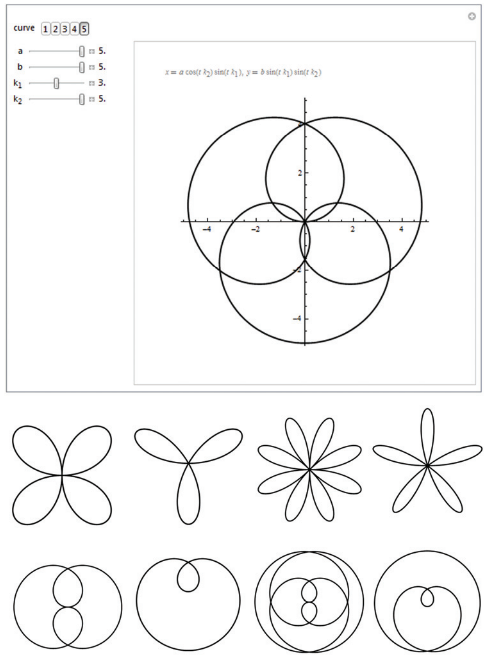

Figure 2 - Demonstration "Looped Curves"

In the first row of illustrations, below demonstration in Figure 2, there is a curve described with equations (10) for various values of parameter k_1, while parameters *a*, *b* and k_2 are constant. The values of parameters are *a=5*, *b=5* and $k_2=1$, while parameter k_1 is equal, respectively

$k_1=2, k_1=3, k_1=4, k_1=5.$ (11)

The second row illustrates the same curve, represented with equations (10), for various values of parameter k_2, and constant values of parameters *a*, *b* and k_1. Values of parameters are *a=5*, *b=5* and $k_1=1$, while parameter k_2 is respectively equal

$k_2=2, k_2=3, k_2=4, k_2=5.$ (12)

[7] Demonstration projects by Milana Dabić, Op. cit.

Demonstration *"Four Space Curves"*, shown in Figure 3, describes parametric equations of circle, Archimedes's spiral, helix and conical spiral. Figure 3 shows demonstration where Archimedes's spiral is chosen:

$$x = at\cos(t), \quad y = at\sin(t), \quad z = c, \quad t \in [0, 2\pi] \tag{13}$$

for parameter values $a=3$ and $c=0$.

Illustrations in the first row, under the demonstration, present a conical spiral with parametric equations

$$x = at\cos(t), \quad y = at\sin(t), \quad z = ct, \quad t \in [0, 2\pi] \tag{14}$$

for parameter c values, respectively

$$c=5,\ c=0 \text{ and } c=-5, \tag{15}$$

while parameter *a* has constant value $(a=5)$.

It is noticeable, by reviewing the illustrations, that Archimedes's spiral is in conical spiral collection too, and it could be described with equations (14) for parameter value $c=0$.

The second row of illustrations in Figure 3 presents a helix with parametric representation

$$x = a\cos(t), \ y = a\sin(t), \ z = ct \tag{16}$$

for constant parameter values *a* and *c* $(a=5, c=1)$, but parameter *t* takes values from interval which has various size. Parameter *t* takes values within the interval, respectively

$$t \in [0, 12\pi], \quad t \in [0, 8\pi], \quad t \in [0, 4\pi]. \tag{17}$$

"Four Surfaces" demonstration shows sphere, ellipsoid, surface eight and hyperboloid, and enables change of parameters which figure in parametric equations. In Figure 4, demonstration shows surface eight with parametric equations:

$$x = a\cos(u)\sin(2v), \quad y = b\sin(u)\sin(2v), \quad z = c\sin(v) \tag{18}$$

for parameter values $a=5$, $b=5$, $c=5$, $u \in [0, 2\pi]$, $v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

The first row of illustrations shows surface for various values of parameter *a*, *b* and *c*:

$$a=2,\ b=2,\ c=5; \quad a=2,\ b=5,\ c=5; \quad a=5,\ b=5,\ c=3; \tag{19}$$

while, interval sizes for *u* and *v* stays the same:

$$u \in [0, 2\pi],\ v \in [-\frac{\pi}{2}, \frac{\pi}{2}]. \tag{20}$$

The second row illustrates change of those intervals, while parameters *a*, *b* and *c* stay constant. For all illustrations $a=5$, $b=5$, $c=5$, while respectively

$$u \in [0, \pi],\ v \in [-\frac{\pi}{2}, \frac{\pi}{2}]; \quad u \in [0, 2\pi],\ v \in [-\frac{\pi}{2}, \frac{\pi}{4}]; \quad u \in [0, \frac{7\pi}{4}],\ v \in [-\frac{\pi}{2}, \frac{\pi}{8}]. \tag{21}$$

CREATING DEMONSTRATIONS

In order to create a demonstration, it is necessary to become familiar with Mathematica's programming language. It is a specific language which provides a great number of functions for mathematics calculus and geometric visualization. Several commands developed within Wolfram Demonstration Project made possible to see the results of some function for various values of its arguments.

There are series of functions for defining and visualizing curves and surfaces. In the source code of previous demonstrations function *ParametricPlot*, which enables visualization of parametric plane curves, and function *ParametricPlot3D*, which enables visualization of parametric space curves and surfaces, are used. Buttons and sliders are implemented with the available functions for demonstration building.

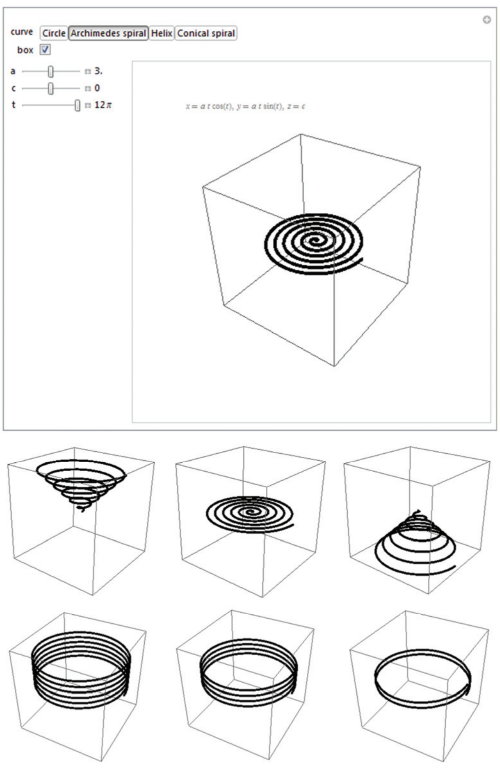

Figure 3 - Demonstration "Four Space Curves"

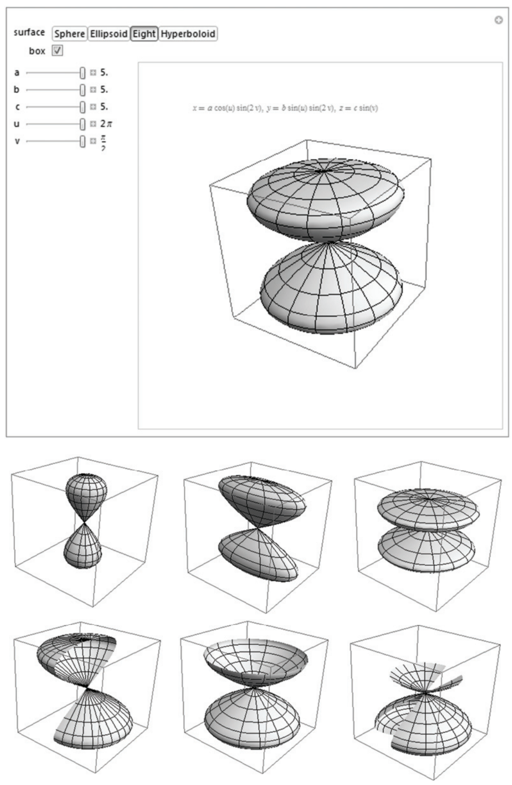

Figure 4 - Demonstration "Four Surfaces"

```
"Eight",
Pane[
Column[{
Text@Row[{Style[Row[{
TraditionalForm[x == a*Cos[u]*Sin[2 v]],", ",
TraditionalForm[y ==b*Sin[u]*Sin[2 v]],",",
TraditionalForm[ z ==c*Sin[v]]}]]}],
ParametricPlot3D[
{aa*Cos[u]*Sin[2 v], bb*Sin[u]*Sin[2 v],cc*Sin[v]},
{u,0,U},{v,-Pi/2,V},
PlotRange→{{-5,5},{-5,5},{-5,5}},
Boxed→ax,Axes→False,ImageSize→300,ImagePadding→20]}],
{400,400},Alignment→{Center,Center}],
```

Figure 5 - Demonstration "Four Surfaces" – part of source code, functions for surface visualization

```
{surface,{"Sphere","Ellipsoid","Eight","Hyperboloid"},
ControlPlacement→Top},
{{aa,3,"a"},1,5,
Appearance→"Labeled",ImageSize→Tiny},
{{bb,1,"b"},1,5,
Enabled→
(surface□"Ellipsoid"||surface□"Eight"||surface□"Hyperboloid"),
Appearance→"Labeled",ImageSize→Tiny},
{{cc,3,"c"},1,5 ,
Enabled→
(surface□"Ellipsoid"||surface□"Eight"||surface□"Hyperboloid"),
Appearance→"Labeled",ImageSize→Tiny},
{{U,2 Pi,"u"},Pi/32,2Pi,Pi/32,
Appearance→"Labeled",ImageSize→Tiny},
{{V,Pi/2,"v"},Pi/2+Pi/32,Pi/2,Pi/32,
Appearance→"Labeled",ImageSize→Tiny},
{{ax,True,"box"},{True,False},ControlPlacement→Top},
```

Figure 6 - Demonstration "Four Surfaces" – part of source code, functions for surface visualization

In Figure 5, a part of source code of the demonstration *"Four Surfaces"* is shown. That part of code implements tab which represents surface *Eight* (Figure 4). In *Traditional Form*, parametric equations are displayed, and with the function *ParametricPlot3D*, surface Eight is visualized. The arguments of function *ParametricPlot3D* are parametric equations, boundaries of intervals within parameters take values and some optional settings for surface visualization.

In Figure 6, some of the functions for creating demonstrations are shown. Part of the source code of the demonstration *"Four Surfaces"* given in Figure 6, implements tabs *Sphere*, *Ellipsoid*, *Eight*, *Hyperboloid* (Figure 4), and shows implementation of sliders which allow change of parameter values *a*, *b*, *c*, *U*, *V*. There is one check box denoted with *box* (Figure 4, and source code in Figure 6). If it's checked, then box around object is displayed. Code also includes some optional functions for sliders and tab placement (*ControlPlacement*), possibility of using them (*Enabled*), their size and appearance…

Before uploading[8], demonstrations are sent for reviewing process. Experts in relevant field check its accuracy and quality, and automated software-quality-assurance methods check its operation. In case of theoretical or technical irregularity, demonstrations are sent back for correction. Therefore, available demonstrations are theoretically correct, they could be counted as academic publications and could be used in educational purposes. Demonstration Project made possible for everyone to use Mathematica visualization and to learn about Mathematica code. With accuracy check and great number of contributors, Project became reliable and extensive source of knowledge.

The Wolfram Demonstration Project continues to grow, because number of contributors and published demonstrations increases daily. Demonstrations are developed in many areas – mathematics, physics, mechanics, biology, art, etc. Although demonstrations mentioned in this paper are developed for architecture generic explorations, Wolfram's team also related them to other areas: 3D graphics, high school mathematics and analytic geometry.

APPLICATION IN THE PROCESS OF ARCHITECTURAL DESIGN

At the Faculty of Architecture, University of

[8] Wolfram Demonstrations Project, Op.cit.

Belgrade, within the series of elective courses named Generic Explorations[9] the authors of this paper made experimental part of architectural form exploration based on plane and space curves and surfaces. Two groups of 20 senior students took part in the experimental design.

The exploration was organized weekly in the following phases: Plane curve study, Generic potential of plane curve, Curve discretization, Space curve study, Space curve materialization, Architectural interpretation of space curve in a given context, Surface study, Curves on surfaces and Final works. Curves and surfaces are defined by parametric equations.

The participants in this experiment easily adopted the presented Demonstration as an explorative tool and stimulated further development of Mathematica demonstrations for exploring curves and surfaces.

These Demonstrations made possible for architecture students to see spatial representation of curves and surfaces, which depends of larger number of parameters. Also, they contributed to examination of generic potential and to application in architectural form exploration. Some of the results of curve studies are shown in Figure 7, and some of the resulted architectural form based on curves and surfaces exploration are shown in Figure 8. Explorations of architecture form are realized in CAAD software, while Mathematica demonstration projects have been considered as a tools for basic precedent explorations.

The funcionality of the Demonstration projects in the process of generic explorations is easy to follow on the example of curve (10) within demonstration project *"Looped Curves"*. Even with systematic mathematical analysis, which include variation of parameters k_1 and k_2 for fixed values of remaining parameter *a* and *b*, one could not predict appearance of curves obtained from some combination of parameter values (Figure 9).

DISCUSION – CURVE AS A FORM GENERATOR

The presented Demonstration projects are mostly related to plane curves for a particular reason. Plane curves have important role in generating architectural geometry. Surfaces are often generated as a trace of movement of a plane curve. That surface contains all those curves, generated by moving the first one, and some of them, often by discretization, could take a role of construction elements. Discretization of surfaces with polygons creates large number of plane curves which could be interpreted like its constructive elements.

Plane curves are very important in generating architecture form. This is justified and proved once more through this experimental work. For this reason the plane curves are at the main focus of this study. Developed Demonstration projects have had an important role in choosing and studying plane curves.

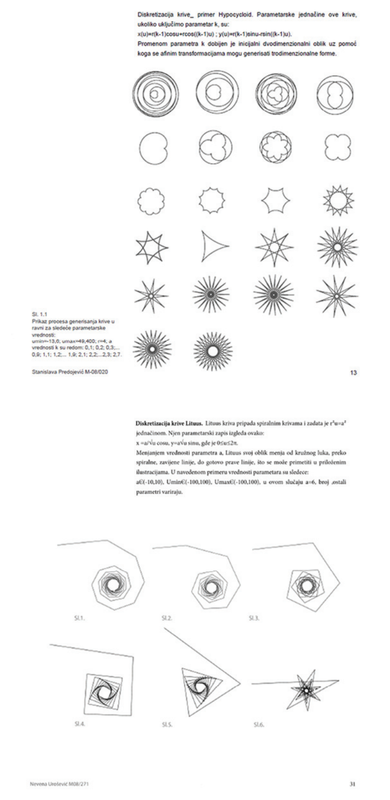

Figure 7 - Plane curve study and Curve discretization

(Student works: Stanislava Predojević, Nevena Urošević)

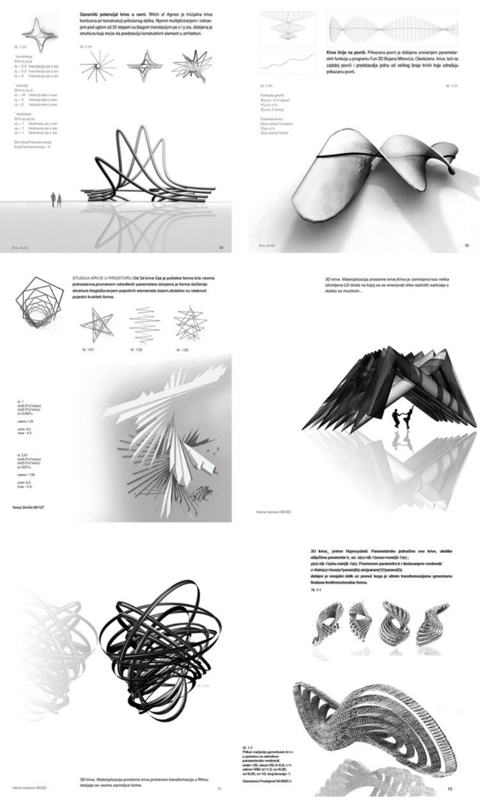

Figure 8 - Generic potential of curves and surfaces –architectural form exploration

(Student works: Ana Jevtić, Sonja Dimčić, Vesna Ivanović, Stanislava Predojević

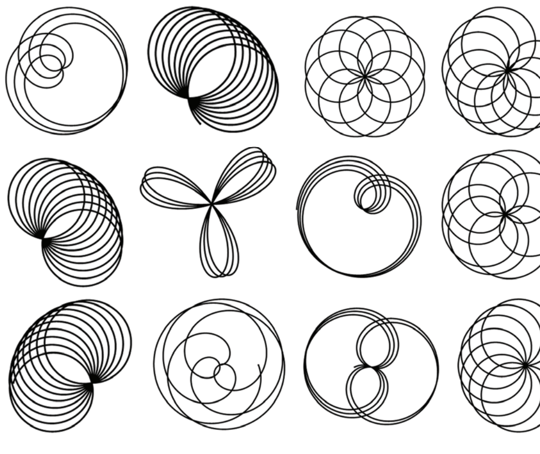

Figure 9 - Study of curve with parametric representation (6)

[9] Web resource: Generic explorations, elective course, Op.cit.

Analyzing the results of the experiments, within Plane curve study phase and Generic potential of plane curve phase it was concluded that curve is often interpreted as architectural element, outlining base geometry of future architectural precedent design (Figure 10). Mathematically defined curve becomes architectural sketch. Advantage of this sketch is that architect can control project much more precisely. Experiments from this study deal mostly with early design stages.

In most cases, as an architect is getting closer to a real architecture form, he deals with project construction. Because of technology that is available today, architects discretize curves and surfaces, according to material performances, project scenario, etc. Students at these courses recognized this potential of discretizated curves and surfaces and they used it in the early design. Moving plane curves in discrete steps, with simultaneous affine transformations, students used them as a generator of initial architectural geometry design (Figure 10).

As mentioned before, plane curve has the main role in generative processes. Also, comparing the results of plane curve exploration and space curve exploration, it turns out that the plane curve has more significant potential in terms of future construction (Figure 8 and Figure 10).

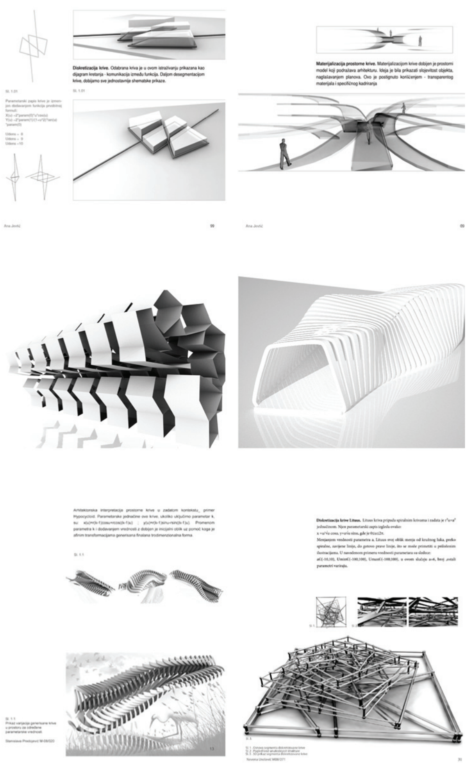

Figure 10 - Curves and surfaces – architectural interpretation (Student works: Ana Jevtić, Stanislava Predojević, Nevena Urošević)

CONCLUSIONS

Presented demonstration projects, have multiple impacts on architectural form exploration. With their help, it is possible to clarify mathematical representations of geometric objects and connection between formulae and graphical interpretation. Also, projects made possible for one to explore collections of curves and surfaces by adding additional parameters in parametric representation. Although demonstrations use Mathematica's programming language which differs from other programming languages, authors of this paper believe that the results are worth the efforts of learning it.

Featuring of these demonstrations[10] shows their mathematical (educational) importance. Their importance as a tool in exploration of architectural form is confirmed within the experimental work on the Faculty of Architecture, University of Belgrade.

The presented demonstration projects had essential part in choice and Study of plane curve, Space curve study and Surface study. They facilitate the understanding of generic potential of curves and surfaces. Demonstrations are developed in an intensive interaction with participants of the experiment as a response to their high criteria and demands. Most of the demonstrations are related to Plane curves because they are significant as architectural form generators.

Demonstrations presented in this paper are included in Wolfram Demonstration Project collection. Although they are developed for the purpose of generic exploration in architecture, they illuminate concept of parametric representation of curves and surfaces and are available within Wolfram Demonstration Project's categories of analytic geometry, high school mathematics and 3D graphics.

References

Bogdanov, A., Manić, B., Petrić, J. (2007), *Pristup proučavanju metoda analize i obnove urbanog tkiva posmatranjem grada kao višestruke fraktalne strukture*, Arhitektura i urbanizam, No. 20-21, pp. 51-60

Devetakovic, M., Petrusevski, Lj., Mitrovic, B., Dabic, M. (2009) *Les Folies Cellulaires – An Exploration in Architectural Design Using Cellular Automata*, XII Generative Art Conference, CD Proceedings, Politechnico di Milano, Italy, pp. 181-192 http://www.generativeart.com/on/cic/GA2009Papers/p18.pdf , accessed 20 Feb. 2010

Filler, A. (2007) *Discovering functional and dynamic aspects of parametric equations by creating computer animations*, Mathematics Education in a Global Community, International Conference, pp. 187-193 http://math.unipa.it/~grim/21_project/21_charlotte_FillerPaperEdit2.pdf, accessed 20 Feb. 2010

Petruševski, Lj., Devetaković M., Mitrović B. (2009) *Self-replicating Systems in Spatial Form Generation – the Concept of Cellular Automata*, SPATIUM International Review, No19, pp. 8-14.

Devetaković, M., Petruševski, Lj., Mitrović, B. (2009) *Arhitektonske interpretacije prostorne forme bazirane na celularnim automatima sa primerima u banjskom kontekstru*, Arhitektura i urbanizam, No. 26, pp. 45-55

Pottmann, H., Asperl A., Hofer M. and Kilian A. (2007) *Architectural Geometry*, Bentley Institute Press

Pottmann H., Brell-Cokcan S. and Wallner J. (2006) *Discrete surfaces for architectural design*, In P. Chenin, T. Lyche and L. L. Schumaker (eds), Curves and Surface Design, Avignon

Shelden D.(2002) *Digital Surface Representation and the Constructibility of Gehry's Architecture*, Thesis (Ph.D.) Massachusetts Institute of Technology, Dept. of Architecture, http://dspace.mit.edu/handle/1721.1/16899?show=full, accessed 10 Feb. 2010

Wolfram, S. (2002) *A New Kind of Science*, Wolfram Media Press, Champaign

Web resources

Demonstration projects by Milana Dabic, http://demonstrations.wolfram.com/author.html?author=Milana+Dabic, accessed 10 Feb. 2010

Generic explorations, http://genericexplorations.blogspot.com/, accessed 10 Feb. 2010

Generic explorations, elective course, Faculty of Architecture, University of Belgrade, AMRES e-Learning system (2009), http://elearning.amres.ac.rs/moodle/course/view.php?id=6a, accessed 10 Feb. 2010

Mathematics in Architecture, Faculty of Architecture, University of Belgrade (2010), http://www.arh.bg.ac.yu/code/navigate.asp?Id=2507, accessed 10 Feb. 2010.

Wolfram Demonstrations Project, http://demonstrations.wolfram.com/, accessed 10 Feb. 2010

[10] Demonstration projects by Milana Dabić, Op. cit.

Received March 2010; accepted April 2010